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A KEY TO THE EXERCISES AND EXAMPLES CONTAINED IN A SCHOOL GEOMETRY

PARTS I–[illegible]

A KEY

TO THE

EXERCISES AND EXAMPLES

CONTAINED IN

A SCHOOL GEOMETRY

PARTS I–VI

BY

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AND

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PREFATORY NOTE

This Key is intended mainly to save the time and lighten the work of Teachers, but it will also serve as a guide to private students reading the subject with few or no opportunities of instruction

Since the latter class of readers may feel some doubt as to the form in which a geometrical solution should be presented, the earlier Exercises are worked out in full, but later it has been thought sufficient in most cases to indicate only the chief steps of the proof

The Answers given in the Geometry to graphical questions were derived from calculation, and are therefore correct as far as the digits are carried But since it is impossible either to *draw* or to *measure* with absolute accuracy, results obtained graphically can only be relied upon as rough approximations Such results may be considered satisfactory if they come within *one per cent* of the formal Answer

Our thanks are due to Mr H C Beaven of Clifton College, for the material help he has given us both in solving Examples and correcting sheets for the press

H S HALL
F H STEVENS

June, 1905

KEY TO SCHOOL GEOMETRY

PART I

Page 13.

3 Let AB, CD intersect at O, and suppose the $\angle$AOD is a rt $\angle$

Then the adjacent $\angle$AOC is also a rt $\angle$ [*Th* 1]

Again, $\angle^s$ COB, BOD are respectively adjacent to $\angle^s$ AOC, AOD,

each is a rt $\angle$ [*Th* 1]

4 The ext angles are supplements of equal angles,

they are equal [*Th* 1 *Cor* 3]

5 The ext angles are supplements of equal angles,

they are equal

6 Let the st line BO make with the st line AOC the adjacent $\angle^s$ AOB, BOC

Suppose OX and OY are the bisectors of these angles respectively

Because $\angle$BOX = half $\angle$BOA,

and $\angle$BOY = half $\angle$BOC,

whole $\angle$XOY = half the sum of $\angle^s$ BOA, BOC

= half of two rt $\angle^s$ [*Th* 1]

= a rt $\angle$

Again, by hypothesis, OY is the external bisector of the $\angle$BOA, and OX is its internal bisector,

the external and internal bisectors of the angle are at rt $\angle^s$ to one another

7. As in Ex 6, the $\angle XOY$ is a rt $\angle$

the $\angle^s$ BOX, BOY are complementary

But $\angle BOX = \angle AOX$, and $\angle BOY = \angle COY$, [*Hyp*]

the $\angle^s$ AOX, COY are complementary

8 The $\angle^s$ COX, XOA are supplementary [*Th* 1]

But $\angle XOA = \angle BOX$, [*Hyp*]

the $\angle^s$ COX, BOX are supplementary

Again, $\angle^s$ AOY, COY are supplementary [*Th* 1]

But $\angle COY = \angle BOY$, [*Hyp*]

the $\angle^s$ AOY, BOY are supplementary

9 $\angle AOB + \angle BOC = 2$ rt $\angle^s$ [*Th* 1]

$= 180°$,

and $\angle AOB = 35°$,

$\angle BOC = 180° - 35° = 145°$,

$\angle COY = \text{half } \angle BOC = 72\frac{1}{2}°$

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1. In one hour the minute-hand makes a complete revolution, *i e* turns through 4 rt $\angle^s$, or 360°

Hence, in 60 minutes the hand turns through 360°

in 1 minute the hand turns through 6°

the angles turned through in 5 minutes, 21 minutes, $43\frac{1}{2}$ minutes, and $14\frac{1}{6}$ minutes are respectively $6° \times 5$, $6° \times 21$, $6° \times 43\frac{1}{2}$ and $6° \times 14\frac{1}{6}$, or 30°, 126°, 261°, and 85°

Again, 6° is turned through in 1 minute

66° is turned through in $\frac{66}{6}$ minutes, *i e* in 11 minutes,

and 222° is turned through in $\frac{222}{6}$ minutes, *i e* in 37 minutes

2 In 12 hours the hour hand turns through 360°,

in 1 hour it turns through 30°

Hence (i) in $3\frac{3}{4}$ hrs it turns through $30° \times 3\frac{3}{4}$, or $112\frac{1}{2}°$,

(ii) in $5\frac{1}{6}$ hrs $30° \times 5\frac{1}{6}$, or 155°

And (iii) the time taken to turn through $172\frac{1}{2}°$ is $\frac{172\frac{1}{2}}{30}$ hours, *i e* $5\frac{3}{4}$ hrs

the time is 5 hrs 45 min

3. The earth turns through 360° every 24 hrs, or 15° every hour,
(i) in $3\frac{1}{3}$ hrs the angle turned through is $15^\circ \times 3\frac{1}{3}$, or 50°,
and (ii) 130° is turned through in $\frac{130}{15}$ hrs, i.e. 8 hrs 40 min

4. (i) The $\angle BOC = 180^\circ - 35^\circ$, or 145°, being the supplement of the $\angle AOC$, [*Th* 1]
the $\angle BOD = 35^\circ$, being equal to the $\angle AOC$, [*Th* 3]
the $\angle DOA = 145^\circ$, being equal to the $\angle BOC$ [*Th* 3]

(ii) The $\angle COB =$ the $\angle AOD$, [*Th* 3]
each is half 250°, i.e. 125°;
the $\angle COA$, being the supplement of the $\angle COB$,
$= 180^\circ - 125^\circ$, or 55°,
the $\angle BOD =$ the $\angle COA = 55^\circ$

(iii) $\angle AOC + \angle COB + \angle BOD = 274^\circ$, [*Hyp*]
and $\angle AOC + \angle COB = 180^\circ$, [*Th* 1]
$\therefore \angle BOD = 274^\circ - 180^\circ = 94^\circ$
the $\angle AOD =$ the supplement of $\angle BOD = 86^\circ$.
Now $\angle AOC = \angle BOD = 94^\circ$, and $\angle BOC = \angle AOD = 86^\circ$

5. The $\angle^s$ BOC, COA are supplementary [*Th* 1]
and $\angle BOC = \angle AOD$ by hypothesis,
$\angle^s$ AOD, AOC are supplementary.
CO, OD are in the same straight line [*Th* 2]

6. Produce XO to Y
Then $\angle AOY = \angle BOX$, and $\angle COY = \angle DOX$ [*Th* 3]
But the $\angle^s$ BOX, DOX are equal by hypothesis,
the $\angle^s$ AOY, COY are equal,
$\therefore$ OY bisects the $\angle AOC$

7. The $\angle^s$ COX, DOX are supplementary [*Th* 1]
The $\angle^s$ DOX, COY are equal, being halves of the equal angles BOD, AOC [*Th* 3]
$\therefore$ the $\angle^s$ COX, COY are supplementary.
OX and OY are in the same straight line [*Th* 2]

8. After folding we have two *equal* angles AOX, BOX with a common arm OX and on the same side of it
the remaining arms OA, OB coincide
If the $\angle AOX$ is greater than the $\angle BOX$, OA falls outside the $\angle XOB$
less " within " "

9 The angles BOC, BOD being both right angles are equal
Hence after folding we have two equal angles BOC, BOD, with a common arm OB and lying on the same side of it, and therefore their remaining arms, viz OD, OC, coincide

10 Unfold the paper and call the crease COD
Then the ∠ˢ AOC, COB (which coincided when the paper was folded) are equal to one another,
and these being adjacent angles, each is a right angle,
that is, CD is perpendicular to AB

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1 Let ABC be the isosceles △, in which AB=AC, and let AX bisect the vert ∠BAC, cutting BC at X

Then in the △ˢ BAX, CAX,

because { BA=CA, by hypothesis,
and AX is common to both,
and ∠BAX=∠CAX, by hypothesis,

the triangles are equal in all respects [*Th* 4]

(i) BX=CX, that is, AX bisects the base
and (ii) ∠AXB=∠AXC,
and these being adjacent angles, each is a right angle,
that is, AX is perpendicular to the base

2 Join PA, PB
Then in the △ˢ POA, POB,

because { OA=OB, by hypothesis,
and OP is common to both,
and ∠POA=∠POB, being right angles,

the triangles are equal in all respects, [*Th* 4]
so that PA=PB

3 In the △ˢ DAB, CBA,

because { DA=CB, being sides of a square,
and AB is common,
and ∠DAB=∠CBA, being rt ∠ˢ,

the △ DAB=the △ CBA in all respects, [*Th* 4]
DB=CA

4 (i) In the $\triangle^s$ LBM, MCN,

because { LB = MC, being halves of equal sides,
and BM = CN, for the same reason,
and $\angle$LBM = $\angle$MCN, being rt. $\angle^s$,

the $\triangle$LBM = the $\triangle$MCN in all respects, [*Th.* 4.]

LM = MN.

(ii) In the $\triangle^s$ ABM, DCM,

because { AB = DC, being sides of a square,
and BM = CM, being halves of BC,
and $\angle$ABM = $\angle$DCM, being rt. $\angle^s$,

the side AM = the side DM. [*Th.* 4.]

(iii) In the $\triangle^s$ ABM, ADN,

because { AB = AD, being sides of a square,
and BM = DN, being halves of equal sides,
and $\angle$ABM = $\angle$ADN, being rt. $\angle^s$,

AM = AN. [*Th.* 4.]

(iv) In the $\triangle^s$ BCN, DCM,

because { BC = DC, being sides of a square,
and CN = CM, being halves of equal sides,
and the angle at C is common to both triangles,

the $\triangle$BCN = the $\triangle$DCM in all respects, [*Th.* 4.]

BN = DM.

5. In the $\triangle^s$ BAY, CAX,

because { BA = CA, being sides of an isos. triangle,
and AY = AX, by hypothesis,
and the angle at A is common to both triangles,

BY = CX. [*Th.* 4.]

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1. (i) Since AB = AD, [*Hyp.*]

the $\triangle$ABD is isosceles,

$\angle$ABD = $\angle$ADB. [*Th.* 5.]

(ii) Similarly it may be shewn that

$\angle$CBD = $\angle$CDB.

(iii) Hence, adding the equal angles in (i) and (ii),

$\angle$ABC = $\angle$ADC.

2 In the $\triangle$ ABC, since AB = AC, by hypothesis,

$\angle$ ABC = $\angle$ ACB [*Th* 5]

And since DB = DC, [*Hyp*]

$\angle$ DBC = $\angle$ DCB [*Th* 5]

Hence the whole $\angle$ ABD = the whole $\angle$ ACD

3 As in the last example it may be shewn that

$\angle$ ABC = $\angle$ ACB,

and $\angle$ DBC = $\angle$ DCB

the remaining $\angle$ ABD = the remaining $\angle$ ACD

4 (i) In the $\triangle^s$ LBM, NCM,

because { LB = NC, being halves of equal sides,
and BM = CM, for a similar reason,
also $\angle$ LBM = $\angle$ NCM, [*Th* 5]

LM = NM [*Th* 4]

(ii) In the $\triangle^s$ LBC, NCB,

because { LB = NC, as above,
BC is common to both,
and $\angle$ LBC = $\angle$ NCB, [*Th* 5]

LC = NB [*Th* 4]

(iii) As in (i), the $\triangle^s$ LBM, NCM are equal in all respects,

$\angle$ BLM = $\angle$ CNM

their supplements, viz $\angle^s$ ALM, ANM are also equal

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1 Let A be the vertex, and BC the base, of an isosceles $\triangle$ ABC, and let BC be bisected at D Join AD

Then in the $\triangle^s$ BDA, CDA,

because { BD = CD, [*Hyp*]
and DA is common to both,
and AB = AC, [*Hyp*]

the triangles are equal in all respects [*Th* 7]

So that (i) $\angle$ BAD = $\angle$ CAD, that is, the $\angle$ BAC is bisected

and (ii) $\angle$ ADB = $\angle$ ADC

And these being adjacent angles, each is a right angle,

that is, AD is perp to BC

2. Let ABCD be a rhombus Join AC

Then in the $\triangle^s$ DAC, BAC,

because $\begin{cases} AD=AB, \text{ being sides of a rhombus}, \\ \text{and } CD=CB, \text{ for the same reason}, \\ \text{and } AC \text{ is common to both triangles}, \end{cases}$

the triangles are equal in all respects [*Th* 7]

So that (i) $\angle ABC=\angle ADC$,

and (ii) $\angle BAC=\angle DAC$, and $\angle BCA=\angle DCA$,

that is, AC bisects each of the $\angle^s$ BAD, BCD

3. Join AC

Then in the $\triangle^s$ ABC, CDA,

because $\begin{cases} AB=CD, \text{ by hypothesis}, \\ \text{and } BC=DA, \quad " \\ \text{and } AC \text{ is common to both}, \end{cases}$

the $\angle$ ABC = the $\angle$ CDA [*Th* 7]

4. (i) Join AD

Then in the $\triangle^s$ ABD, ACD,

because $\begin{cases} AB=AC, \text{ by hypothesis}, \\ \text{and } BD=CD, \quad " \\ \text{and } AD \text{ is common to both}, \end{cases}$

$\angle$ ABD = the $\angle$ ACD [*Th.* 7]

(ii) The second case is similarly proved

5 In the $\triangle^s$ BAD, CAD,

because $\begin{cases} BA=CA, \text{ by hypothesis}, \\ \text{and } AD \text{ is common to both}, \\ \text{also } BD=CD, \end{cases}$

the $\angle$ BAD = the $\angle$ CAD, [*Th* 7]

and the $\angle$ BDA = the $\angle$ CDA

6. Let ABC be an isosceles triangle, and X, Y the middle points of the equal sides AC, AB Join BX, CY

Then in the $\triangle^s$ YBC, XCB,

because $\begin{cases} YB=XC, \text{ being halves of equal sides}, \\ \text{and } BC \text{ is common to both}, \\ \text{also } \angle YBC=\angle XCB, \quad [Th. 5] \end{cases}$

YC = XB [*Th* 4]

7. In the base BC of the isosceles $\triangle$ ABC, let D and E be two points such that BD = CE. Join AD, AE.

Then in the $\triangle^s$ ABD, ACE,

because AB = AC, [Hyp.]
BD = CE, [Hyp.]
and $\angle$ ABD = $\angle$ ACE, [Th. 5]
AD = AE [Th. 4]

8. In the equilateral $\triangle$ ABC, let D, E, F be the middle points of the sides BC, CA, AB, respectively. Join DE, EF, FD.

Then in the $\triangle^s$ FAE, FBD,

because FA = FB, [Hyp.]
and AE = BD, being halves of equal sides,
and $\angle$ FAE = $\angle$ FBD, [Th. 5, Cor. 2]
FE = FD [Th. 4]

Similarly FD = DE, the $\triangle$ FDE is equilateral.

9. Let ABC be an isosceles triangle, and let BO, CO bisect the base $\angle^s$ ABC, ACB.

(i) Then the $\angle$ ABC = the $\angle$ ACB, [Th. 5]
the $\angle$ OBC = the $\angle$ OCB, being halves of equal angles,
OB = OC [Th. 6]

(ii) Again in the $\triangle^s$ ABO, ACO,

because AB = AC, [Hyp.]
and BO = CO, by (i),
and $\angle$ ABO = $\angle$ ACO, being halves of equal angles,
the $\angle$ BAO = the $\angle$ CAO [Th. 4]

10. Let the diagonals of the rhombus ABCD intersect at X. Then, as in Ex. 2, the $\triangle^s$ BAC, DAC may be shewn equal in all respects by Theor. 7. the $\angle$ BAC = the $\angle$ DAC.

Then in the $\triangle^s$ BAX, DAX,

because BA = DA, [Hyp.]
and AX is common,
and $\angle$ BAX = $\angle$ DAX,
. BX = DX, and the $\angle$ AXB = the $\angle$ AXD [Th. 4]
and these angles, being adjacent, are rt. $\angle^s$.

11. In the Δ^s FAB, EAC,

because $\begin{cases} FA = EA, & [Hyp] \\ \text{and } AB = AC, & \\ \text{and } \angle FAB = \angle EAC, & [Th\ 3] \end{cases}$

$FB = EC$ [*Th* 4]

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3. Theoretically by Theorem 4 Experimentally by drawing one of the triangles on tracing paper and superposing it on the other

4. If the drawing and measurements are accurate, the values *found* for b, c, and A in the second triangle should agree with the values *given* in the first triangle

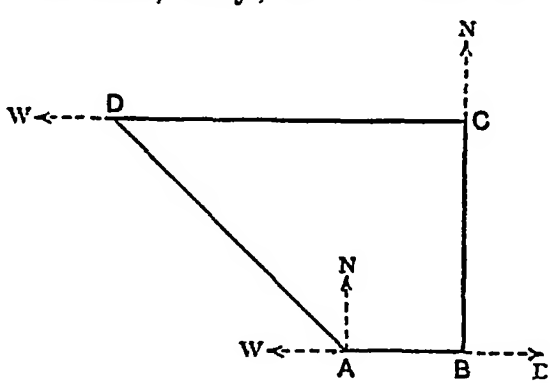
Hence we conclude that a triangle is completely determined from either of the following data:

(i) two sides and the included angle, [Compare

(ii) two angles and the side which joins them *Th* 4, 17]

7. Draw a horizontal line AB, 3 0″ long, to represent the distance between the foot of the pole B and the extremity of the shadow A At A make an angle BAD = 42° From B draw BC perp to AB and meeting AD in C Then C represents the position of the top of the pole By measurement BC = 2 7″, and therefore the height of the pole is 27 ft

8. The figure is drawn on one-third the given scale If drawn on the required scale, it will be found by measurement that DA = 4 24″, nearly; and the ∠ DAB = 135°

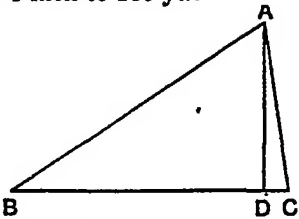


DA represents 424 yards, and since AD bisects the angle between AN and AW, the bearing of D from A is N W

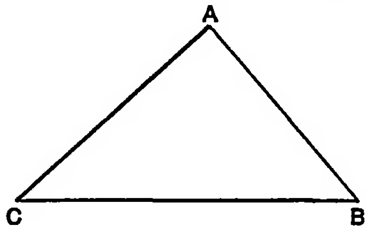
9 A convenient scale to use is that of 1 inch to 100 yds [The diagram is given to half this scale] It will then be found by measurement that

AB = 2·81″, AC = 1·55″,

and AD (drawn from A perp to BC) = 1·53″ Hence the distances represented by these lines are 281 yds, 155 yds, and 153 yds respectively



10 A convenient scale is that in which 1 inch represents 100 yds Using this, draw CB of length 3·2″, and with the protractor make ∠BCA = 42° Cut off CA = 2·45″, and join AB [The diagram is given to half this scale] Then AB represents the distance between the two places By measurement on the figure, AB is found to be 2·14″, nearly, hence the distance between the two places is 214 yds



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1 Take any point D in the base BC and join AD
Then in the △ ABD, int ∠ABD < ext ∠ADC, [*Th* 8]
and in the △ ADC, int ∠ACD < ext ∠ADB [*Th* 8]
the ∠s ABD, ACD are together less than the ∠s ADC, ADB,
that is, less than two right angles [*Th* 1]

2 (i) Produce BD to meet AC in K
Then in the △ DKC, ext ∠BDC > int opp ∠DKC, [*Th* 8]
and in the △ AKB, ext ∠DKC > int opp ∠BAC [*Th* 8]
Still more then is the ∠BDC > the ∠BAC

(ii) Produce AD to meet BC at X
Then in the △ BAD, ext ∠BDX > int opp ∠BAD, [*Th* 8]
and in the △ DAC, ext ∠XDC > int opp ∠DAC [*Th* 8]
adding these results, the whole ∠BDC > the ∠BAC

3. In the $\triangle$ ABC let BC be produced to D and CB to E

Then the $\angle^s$ ABE, ABC together $=2$ rt $\angle^s$; [*Th* 1]

also the $\angle^s$ ACD, ACB together $=2$ rt $\angle^s$;

hence the $\angle^s$ ABE, ABC, ACD, ACB $=4$ rt $\angle^s$,

and of these, the $\angle^s$ ABC, ACB are together less than 2 rt $\angle^s$, [*Th* 8, *Cor* 1]

$\therefore$ the $\angle^s$ ABE, ACD are together greater than 2 rt $\angle^s$

4. Let ABC be the given line and O the given point outside it and let OA, OB, OC be three lines, supposed all equal, drawn from O to the line

Then because OA$=$OC, $\angle$OAC$=\angle$OCA; [*Th* 5]

and because OA$=$OB; $\angle$OAB$=\angle$OBA, [*Th* 5]

hence $\angle$OCA$=\angle$OBA But this is impossible by Theor 8, for one of these angles is an ext angle of the $\triangle$OBC and the other its int opp angle

$\therefore$ more than two equal straight lines cannot be drawn from O to AB

5 In an isosceles triangle each of the *equal* angles is acute, for by Theor 8, Cor 1 their sum is less than two rt angles

each of the ext $\angle^s$ at the base is the supplement of an acute angle, and is consequently obtuse

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1. From Theor 8, Cor 2 it follows that every triangle must have *at least* two acute angles, the rt angle is the greatest angle; the hypotenuse, which is opposite to the rt angle, is the greatest side [*Th* 10]

2. Let ABC be a triangle and BC its greatest side Then, by Theor 9, since BC is greater than either AB or AC we have the $\angle$BAC greater than either the $\angle$BCA or the $\angle$ABC Thus if either of these angles is obtuse, the $\angle$BAC is also obtuse, and we have two angles of a triangle whose sum is greater than two rt angles; which is impossible by Theor 8, Cor 1 Hence each of the angles ABC, BCA is acute

3 Let ABC be the triangle and BD, DC the lines drawn from B and C to a point D within the triangle Produce BD to meet AC in X

From the $\triangle$ DXC, DX+XC>DC [*Th* 11]
Add BD to each side, then BX+XC>BD+DC
Also from the $\triangle$ BAX, BA+AX>BX
Add XC to each side, then BA+AC>BX+XC
Hence BA+AC>BD+DC

4 Since AB=AC, $\angle$ABC=$\angle$ACB [*Th* 5]
But in the $\triangle$ ACD, ext $\angle$ACB>int opp $\angle$ADB, [*Th* 8]
the $\angle$ABD>the $\angle$ADB,
AD>AB [*Th* 10]

5 Let ABCD be a quadrilateral having AB the least side, and CD the greatest Join BD
Then since DA>AB [*Hyp*], $\angle$ABD>$\angle$ADB, [*Th* 9]
and since DC>BC [*Hyp*], $\angle$DBC>$\angle$BDC,
the whole $\angle$ABC>the whole $\angle$ADC
So, by joining AC, the $\angle$BAD may be shewn>the $\angle$BCD

6 Let AX be drawn to meet the base in X
Then since AC is not greater than AB,
AC is equal to, or less than AB,
the $\angle$ABC is equal to, or less than, the $\angle$ACB [*Th* 5 and 9]
But ext $\angle$AXB is greater than int opp $\angle$ACB, [*Th* 8]
the $\angle$AXB is greater than the $\angle$ABX,
AB is greater than AX [*Th* 10]

7 Let AB be>AC, then the $\angle$ACB>the $\angle$ABC, [*Th* 9]
hence the $\angle$OCB>the $\angle$OBC, OB>OC [*Th* 10]

8 In the $\triangle$ ABC, BC is less than the sum of BA and AC [*Th* 11]
Take AC from each of these unequals, then the difference of BC and AC is less than BA

9 Let ABC be the triangle, and O any point
In the $\triangle$ OAB, OA+OB>AB, [*Th* 11]
in the $\triangle$ OBC, OB+OC>BC, add these results,
in the $\triangle$ OCA, OC+OA>CA,
twice the sum of OA, OB, OC>the sum of AB, BC, CA,
i e the sum of OA, OB, OC>half the perim of the $\triangle$ABC

10 Let AC, BD be the diagonals of the quad^l ABCD.

Then from the $\triangle$ ABC, $AB+BC>AC$, [*Th* 11]

and from the $\triangle$ ADC, $AD+DC>AC$,

by addition, the perimeter > *twice* AC

Similarly, the perimeter > *twice* BD

, by addition. *twice* perimeter > *twice* sum of diagonals

$\therefore$ perimeter > sum of diagonals

11. In the $\triangle$ ABC, let AX bisect the $\angle$ BAC and meet BC at X

Then ext $\angle AXC >$ int opp $\angle BAX$, [*Th* 8]

$\therefore$ $\angle AXC$ is also $> \angle CAX$; [*Hyp*]

$AC > XC$ [*Th* 10]

Similarly, $AB > BX$;

$\therefore$, by addition, AB, AC together > BC

12. Let O be any point within the $\triangle$ ABC Produce AO to meet BC in X

Then $AC+CX>AX$ [*Th* 11] $AC+CB>AX+XB$

But $OX+XB>OB$ [*Th* 11] $AX+XB>AO+OB$

$$\left.\begin{aligned} \text{Hence } AO+BO &< AC+BC, \\ \text{similarly } BO+CO &< BA+CA, \\ \text{and } CO+AO &< CB+AB; \end{aligned}\right\}$$

$\therefore$, by addition, *twice* the sum of OA, OB OC is less than *twice* the sum of AB BC. CA,

or $OA+OB+OC<AB+BC+CA$

13. Let O be the given pt, AC and BD the diagonals,

then $AO+OC>AC$, [*Th* 11]

and $BO+OD>BD$; [*Th* 11]

$\therefore$, by addition $OA+OB+OC+OD>AC+BD$.

The exceptional case is when O is at the intersection of the diagonals

14. Let ABC be the triangle, AD the median bisecting BC Produce AD to E, making $DE=AD$ Join EC

Then, by Theorem 4, we can prove the $\triangle^s$ ADB, EDC equal in all respects, and $\therefore$ $AB=CE$

But $AC+CE>AE$, [*Th.* 11.]

$\therefore$ $AC+AB>2AD$.

15 With the Figure of last example let BF, CH be the medians bisecting AC, AB respectively

Then, as in Ex. 14, we can prove $AB+AC>2AD$
similarly $BC+BA>2BF$,
and $CA+CB>2CH$

∴ by addition, twice sum of the sides > twice sum of the medians, that is, the sum of the medians is less than the perimeter

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1 Since ∠EGB=55°, its supplementary angle BGH=125° [*Th* 1]
Because AB, CD are par[l], ∠GHC=alt ∠BGH [*Th* 14]
∠GHC=125°
Because AB, PQ are par[l], int ∠HKQ=ext. ∠EGB. [*Th* 14]
∠HKQ=55°
∠QKF, being supplementary to ∠HKQ is 125° [*Th* 1]

2 Let each of the st lines AB, CD be perp to a st line which meets them at P and Q.
Then the ∠s APQ, PQD are equal being rt angles and since these angles are alternate, AB and CD are parallel [*Th* 13]

3 Let PQ meet the parallels AB, CD at P and Q, and let PQ be perp to AB
Then since AB and CD are par[l], and PQ meets them,
the ∠PQD=the alt. ∠APQ [*Th* 14]
=a rt. angle [*Hyp*]

4 Let ABC be any angle, and PQR an angle formed by two lines PQ, QR par[l] to AB, BC respectively
First, when the ∠s ABC, PQR are either both acute or both obtuse
Let PQ, produced if necessary, cut BC at X.
Then since AB and PQ are par[l], and BXC meets them,
∠PXC=corresponding ∠ABC [*Th* 14]
But since BC and QR are par[l], and PXQ meets them;
∠PXC=corresponding ∠PQR [*Th* 14]
∠ABC=∠PQR

Next, when the angles ABC, PQR are one acute and one obtuse
Produce PQ to meet BC at S Then the $\angle^s$ ABC, RQS are either both acute or both obtuse
, as in the first case, $\angle ABC = \angle RQS$
But $\angle^s$ RQS, PQR are supplementary [*Th* 1]
$\angle^s$ ABC, PQR are supplementary

5. In the $\triangle^s$ AOC, BOD,

$$\text{because}\begin{cases} \left.\begin{array}{r} AO = BO, \\ \text{and } OC = OD, \end{array}\right\} \text{by hypothesis}, \\ \text{and } \angle AOC = \angle BOD, \text{ being vert opp } \angle^s, \end{cases} \quad [Th\ 3]$$

the $\angle OAC$ = the $\angle OBD$, [*Th* 4]
and these are alternate angles, AC and BD are parallel [*Th* 13]

6 Let XY cut the equal sides AB, AC of the isosceles $\triangle$ ABC at X and Y
Then since XY, BC are par[1], and AXB meets them,
ext $\angle AXY$ = int opp $\angle ABC$ [*Th* 14]
Similarly the $\angle AYX$ = the $\angle ACB$
But the $\angle ABC$ = the $\angle ACB$ [*Th* 5], the $\angle AXY$ = the $\angle AYX$

7. Let P be any point in the bisector of the $\angle AOB$, and let PQ be drawn par[1], to OB
Then since OB, PQ are par[1], and OP meets them,
$\angle QPO$ = alt $\angle POB$ [*Th* 14]
= $\angle POQ$, [*Hyp*]
QO = QP [*Th* 6]

8. Let AD be the bisector of the $\angle BAC$
Then the $\triangle^s$ BAD, CAD can be proved equal in all respects by Theor 4, and $\angle ADB = \angle ADC$, that is, AD is perp to BC
Thus AD and ZYX, being both perp to BC, are parallel,

$$\left.\begin{array}{r} \angle BAD = \text{alt } \angle AYZ, \\ \text{also } \angle CAD = \text{int opp } \angle AZY, \end{array}\right\} \quad [Th\ 14]$$

$\angle AYZ = \angle AZY$ AY = AZ [*Th* 6]

9. In the $\triangle$ ABC, let BA be produced to D, and let AX bisect the $\angle CAD$, and be par[1] to BC

$$\left.\begin{array}{r} \text{Then ext } \angle XAD = \text{int opp } \angle CBA, \\ \text{and } \angle XAC = \text{alt } \angle BCA, \end{array}\right\} \quad [Hyp \text{ and } Th\ 14]$$

but $\angle XAD = \angle XAC$ [*Hyp*], $\angle CBA = \angle BCA$,
AB = AC [*Th* 6]

10 From O, any pt on the bisector of $\angle$BAC, draw OP par[l] to AB and OQ par[l] to AC

Then, by Theor 14, $\angle$s POA, QOA are equal to $\angle$s QAO, PAO respectively

But $\angle$PAO $=$ $\angle$QAO [*Hyp*]

the four angles are all equal

Now imagine the $\triangle$ AOQ folded over about AO Then because $\angle$QAO $=$ $\angle$PAO, the line AQ coincides in direction with the line AP

Similarly OQ coincides in direction with OP

Q, the common pt of AQ and QO, must coincide with P the common pt of AP and PO

the $\triangle$s AOP, AOQ coincide, and AQ $=$ AP, and OP $=$ OQ

But $\angle$PAO $=$ $\angle$POA, [*Proved*] AP $=$ PO [*Th* 6]

Hence the four sides AP, PO, OQ, QA are all equal Thus the figure is equilateral, and is therefore a rhombus

[*Exceptional case* If the given angle is a right angle, the resulting figure is a square]

11 Let D be the pt of intersection of AB and CD,

then $\angle$XYD $=$ alt $\angle$YDA $=$ $\angle$YDX [*Th* 14, and *Hyp*]

XY $=$ XD [*Th* 6]

Similarly XD $=$ XZ,

YX $=$ XZ

12 The lines PA, QB turn through 72°, 60° respectively per sec

Hence the line PA turns through 12° more per sec than QB

Now when PA is again parallel to QB, and pointing in the opposite direction, it must have turned through 180° more than QB

Hence the time taken in this case is $\frac{180}{12}$ or 15 seconds

Again, when PA and QB are both pointing the same way the angle gained by PA is 360°

Thus the time taken is $\frac{360}{12}$ or 30 seconds

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5 In the Fig of Theor 16 we have $\angle$ACD $=$ $\angle$BAC $+$ $\angle$ABC [*Th* 16]

$\angle$ABC $=$ $\angle$ACD $-$ $\angle$BAC $=$ 134° $-$ 42° $=$ 92°,

and $\angle$ACB $=$ supplement of $\angle$ACD $=$ 46° [*Th* 1]

6 $\angle ACD = \angle BAC + \angle ABC$, [*Th* 16]

$\angle BAC = \angle ACD - \angle ABC = 118° - 51° = 67°$,

also $\angle ACB$ = supplement of $\angle ACD = 180° - 118° = 62°$

7. Let AD be drawn through the vertex A par[1] to base BC

Then $\angle BCA$ = alt $\angle CAD$ [*Th* 14]

the three $\angle^s$ of $\triangle ABC = \angle^s$ CBA, BAD

= 2 rt $\angle^s$ [*Th* 14]

8 Let the lines XP, XQ be respectively perp to the lines AP, AQ, which form an acute angle at A Let PX, produced if necessary, meet AQ in O

Then $\angle QXO$ = compt of $\angle QOX$ [*Th* 16 *Inf* 3]

= compt of $\angle POA$

$= \angle PAO$

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1 Here $B = 2A$, $C = 3A$, $A + 2A + 3A = 180°$

$A = 30°$, and $B = 60°$, $C = 90°$

2 Let A be the vertical angle, B and C the base angles

(i) Here $B = C = 2A$, $A + 2A + 2A = 180°$

$A = 36°$, and $B = C = 72°$

(ii) Here $B = C = 4A$, $A + 4A + 4A = 180°$

$A = 20°$, and $B = C = 80°$

3 To construct the triangle, draw any line AB at a pt P in AB make $\angle APK = 94°$, and at any pt Q in BP make $\angle BQL = 126°$ Then if PK and QL cut at R, PQR is the required triangle The interior angles at P, Q are the supplements of 94° and 126° respectively, viz. 86° and 54° Hence $R = 180° - (86° + 54°) = 40°$

4 Here the vertical angle $A = 180° - 162° = 18°$

also $B + C = 162°$, $B - C = 60°$, whence $B = 111°$, $C = 51°$

5 Let the bisectors of the base angles B and C meet at K

Then $A = 180° - (B + C) = 180° - (84° + 62°) = 34°$

Also $\angle KBC = 42°$, and $\angle KCB = 31°$,

$\angle BKC = 180° - (42° + 31°) = 107°$

6 Let AB, AC be produced to P and Q respectively, and the bisectors of the $\angle^s$ CBP, BCQ meet at K

Then $\angle PBC = 180° - 74° = 106°$, and $\angle KBC = 53°$

Also $\angle QCB = 180° - 62° = 118°$, and $\angle KCB = 59°$

$\angle BKC = 180° - (53° + 59°) = 68°$

7 Suppose the fourth angle contains x degrees

Then $114\frac{1}{2}° + 50° + 75\frac{1}{2}° + x° = 4$ rt $\angle^s$ [*Th* 16 *Inf* 5]

$x° + 240° = 360°$, $x = 120$

8 The sum of the four angles of a quad' can be shewn to be 4 rt $\angle^s$ (either by Theor 16, Cor 1, or directly by drawing a diagonal)

Hence $A + 2A + 3A + 4A = 360°$, whence $A = 36°$,

and $B = 72°$, $C = 108°$, $D = 144°$

9 Here $40° + 78° + 122° + 135° + x° + 4$ rt $\angle^s = 10$ rt $\angle^s$ [*Th* 16 *Cor* 1]

$x° + 375° = 6$ rt $\angle^s$, and $x = 165$

10 (i) The n equal angles $+ 4$ rt $\angle^s = 2n$ rt $\angle^s$ [*Th* 16 *Cor* 1]

the n angles $= (2n - 4)$ rt. $\angle^s$

each angle $= \dfrac{2n-4}{n}$ rt. $\angle^s$

(ii) Each of the $(n - 2)$ triangles so formed contains 2 rt. $\angle^s$ [*Th* 16]

the sum of all their angles $= 2(n - 2)$ rt $\angle^s$

But these angles make up the n equal angles of the polygon

the n angles $= 2(n - 2)$ rt $\angle^s$

each angle $= \dfrac{2(n-2)}{n}$ rt $\angle^s$

11 Let n denote the number of sides of the polygon

Then (i) $108° \times n + 360° = 180° \times n$, whence $n = 5$ [*Th* 16 *Cor* 1]

(ii) $156° \times n + 360° = 180° \times n$, whence $n = 15$ [*Th* 16 *Cor* 1]

12 For this to be possible 4 rt $\angle^s$ must be exactly divisible by the number of rt angles in each angle of the figure

Hence if n denote the number of sides, 4 must be exactly divisible by $\dfrac{2(n-2)}{n}$ [See Ex. 10]

$$\frac{4n}{2(n-2)}, \text{ i e } \frac{2n}{n-2}, \text{ is an integer}$$

Since this can be written $2+\frac{4}{n-2}$, we see that $\frac{4}{n-2}$ is an integer, and the only possible values of n are found by trial to be 3, 4, 6

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1 The sum of the 6 ext. $\angle^s = 360°$ [*Th* 16 *Cor* 2]

each ext. $\angle = 60° =$ int $\angle$ of an equilateral $\triangle$

2 Each ext $\angle$ of regular octagon $= \frac{1}{8}$th of 4 rt $\angle^s = 45°$

Each ext $\angle$ of regular decagon $= \frac{1}{10}$th of 4 rt $\angle^s = 36°$

3 The number of sides in (i) is $\frac{360}{30}$, i e 12, and in (ii) $\frac{360}{24}$, i e 15

4 Let PHKQ cut the two par^l lines AB, CD at H, K Let HX, KX be the bisectors of $\angle^s$ BHK, HKD

Then $\angle BHK + \angle HKD = 180°$ [*Th* 14]

$\angle XHK + \angle XKH = 90°$, [*Hyp*]

the third angle of $\triangle$ HKX $= 90°$

5 Let ABC be the triangle with AB, AC produced to D and E

$\angle DBC = A + C$, [*Th* 16]

$\angle ECB = A + B$,

$\angle DBC + \angle ECB = (A+B+C) + A = 180° + A$

6 $\angle BOC = 180° - \angle OBC - \angle OCB = 180° - \frac{1}{2}B - \frac{1}{2}C$

$$= 180° - \tfrac{1}{2}(B+C)$$
$$= 180° - \tfrac{1}{2}(180° - A)$$
$$= 90° + \frac{A}{2}$$

7 $\angle BOC = 180° - \angle OBC - \angle OCB = 180° - \frac{1}{2}(180° - B) - \frac{1}{2}(180° - C)$

$$= \tfrac{1}{2}(B+C)$$
$$= \tfrac{1}{2}(180° - A)$$
$$= 90° - \frac{A}{2}$$

8 Let ABCD be the quad^l, and let the bisectors of A and B meet at O

Then $\angle AOB = 180° - \frac{1}{2}A - \frac{1}{2}B$

$= \frac{1}{2}\{360° - (A+B)\}$

$= \frac{1}{2}(C+D)$, for $A+B+C+D=360°$ [*Th* 16 *Inf* 5]

9 $\angle ACD = \angle ADC$, and $\angle ACB = \angle ABC$ [*Th* 5]

$\angle BCD = \angle DBC + \angle BDC$, $\angle BCD$ is a rt $\angle$ [*Th* 16 *Inf* 4]

10 Let the $\triangle$ ABC be rt angled at C, and let D be the mid-point of AB

Produce CD to E, making DE=CD, join BE Then, as in Theor 8, the $\triangle^s$ BDE, ADC are equal in all respects

$\angle EBD = \angle DAC$, and EB=AC

Hence $\angle EBC = \angle DAC + \angle ABC$

= a rt $\angle$, since ACB is a rt $\angle$ [*Th* 16]

Thus in the $\triangle^s$ EBC, ACB,

because $\begin{cases} EB = AC, \\ \text{and BC is common,} \\ \text{and } \angle EBC = \angle ACB \text{ being rt } \angle^s, \end{cases}$

EC=AB [*Th* 4]

But DC is half EC, by construction DC is half AB

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1 In the isosceles $\triangle$ ABC, let BX, CY be drawn perp respectively to the equal sides AC, AB

Then in the $\triangle^s$ YBC, XCB,

because $\begin{cases} \angle YBC = \angle XCB, \quad [\textit{Th } 5] \\ \text{and } \angle BYC = \angle CXB, \text{ being rt } \angle^s, \\ \text{and BC is common to both,} \end{cases}$

CY=BX [*Th* 17]

2 Let O be any point on the bisector of the $\angle BAC$

Draw OP, OQ perp respectively to AB, AC

Then in the $\triangle^s$ OAP, OAQ.

because $\begin{cases} \angle OAP = \angle OAQ, \quad [\textit{Hyp}] \\ \text{and } \angle OPA = \angle OQA, \text{ being rt } \angle^s, \\ \text{and OA is common to both,} \end{cases}$

OP=OQ. [*Th* 17]

3. In the $\triangle$s AOX, BOY,

because { $\angle$AOX = $\angle$BOY, being vert. opp. $\angle$s, [*Th.* 3.]
and $\angle$AXO = $\angle$BYO, being rt. $\angle$s,
and AO = BO, [*Hyp.*]

AX = BY. [*Th.* 17.]

4. In the $\triangle$ ABC, let AD, the bisector of the $\angle$BAC, be perp. to BC.

Then in the $\triangle$s BAD, CAD,

because { $\angle$BAD = $\angle$CAD, [*Hyp.*]
$\angle$BDA = $\angle$CDA, being rt. $\angle$s, [*Hyp.*]
and AD is common to both,

AB = AC. [*Th.* 17.]

5. In the $\triangle$ ABC let the perp. AD bisect the base BC.

Then in the $\triangle$s ADB, ADC,

because { DB = DC, [*Hyp.*]
and AD is common to both,
and $\angle$BDA = $\angle$CDA, being rt. $\angle$s,

AB = AC. [*Th.* 4.]

6. Let the base BC be bisected by AD, which also bisects the vert. $\angle$BAC.

Produce AD to E, making DE equal to DA, and join CE.

Then in the $\triangle$s ADB, EDC,

because { DB = DC, [*Hyp.*]
DA = DE,
also $\angle$ADB = $\angle$EDC, [*Th.* 3.]

AB = EC,

and $\angle$BAD = $\angle$CED. [*Th.* 4.]

$\angle$CED = $\angle$CAD,

AC = EC. AB = AC. [*Th.* 5.]

7. Let POQ, terminated by the given par^l st. lines at P and Q, be bisected at O.

Through O draw XOY perp. to the par^l lines.

Then in the $\triangle$s PXO, QYO,

because { $\angle$PXO = $\angle$QYO, being rt. $\angle$s,
and $\angle$POX = $\angle$QOY, [*Th.* 3.]
and OP = OQ, [*Hyp.*]

OX = OY. [*Th.* 17.]

8 Let the st line AOB be bisected at O and terminated at A, B by the par' lines AP, QB Let POQ be any other st line drawn through O and terminated by the parallels

Then in the $\triangle^s$ AOP, BOQ,

because $\left\{\begin{array}{l} \angle POA = \angle QOB, \\ \text{and } \angle PAO = \text{alt } \angle QBO, \\ \text{and } OA = OB, \end{array}\right.$ [Th 3] [Th 14] [Hyp]

OP = OQ [Th 17]

9 Let AB, CD be the par' st lines, O the pt equidistant from them, and AOD, BOC the st lines drawn through O and terminated by AB, CD

Through O draw XOY perp to AB, CD and meeting them in X and Y respectively

Then in the $\triangle^s$ AOX, DOY,

because $\left\{\begin{array}{l} \angle AXO = \angle DYO, \text{ being rt } \angle^s, \\ \text{and } \angle AOX = \angle DOY, \\ \text{and } OX = OY, \end{array}\right.$ [Th 3] [Hyp]

AX = DY [Th 17]

Similarly BX = CY

, by addition (or subtraction), AB = CD

10 In the $\triangle^s$ ABC, ADC,

because $\left\{\begin{array}{l} AB = AD, \\ BC = DC, \\ \text{and } AC \text{ is common}, \end{array}\right.$ [Hyp]

$\angle$BAC = $\angle$DAC, and $\angle$BCA = $\angle$DCA [Th 7]

Join BD, cutting AC in K

Then in $\triangle^s$ BAK, DAK, we have BA = DA, and AK common, and $\angle$BAK = $\angle$DAK [Proved]

the angles AKB, AKD are equal [Th 4]

AC is perp to BD

11 In the $\triangle^s$ AOB, COD,

because $\left\{\begin{array}{l} OA = OC, \\ \text{and } \angle OAB = \angle OCD, \text{ being rt } \angle^s, \\ \text{and } \angle AOB = \text{vert. opp } \angle COD, \end{array}\right.$ [Const] [Th 3]

AB = CD [Th 17]

That is, CD measures the width of the river

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1. (i) See Theor 16 and Cor 1
   (ii) See Theor 16 Cor 2

2. See p 17 Def 8

3 See Theorems 5 and 9
By Theor 5, the $\angle$A = the $\angle$C, and each is acute
Again since b is less than each of the other sides, the $\angle$B is less than each of the other angles [*Th* 9], ∴ all the three angles are acute

4 The Theorems are 6 and 10
(i) $C = 180° - (48° + 51°) = 81°$
Thus C being the greatest angle, c will be the greatest side [*Th* 10]
(ii) $C = 180° - (62\frac{1}{2}° + 62\frac{1}{2}°) = 55°$
Thus $\angle$A being greater than $\angle$C we have a is greater than c [*Th* 10]
And $\angle A = \angle B$, ∴ $a = b$ [*Th* 6]
Hence a, b are equal, and each greater than c

5 (i) equal, by Theor 17
(ii) equal, by Theor 4
(iii) not necessarily equal, see (i), p 50
(iv) equal, by Theor 7
(v) ambiguous, see (ii), p 50
(vi) equal, by Theor 18

6 See pp 50, 51

7. It will be noticed that in each of the four theorems which prove two triangles congruent (Theors 4, 7, 17, 18) we have given *three* parts of one triangle equal to three parts of the second triangle, and that these three relations are independent, *i e* no one of them is simply a consequence of the existence of the other two This may be summarised by saying that in order to prove two triangles congruent we must have *at least* three *independent* relations between their several parts [N.B The existence of *any* three such relations does not prove congruence See (ii), p. 50]

Now in the case given us the three relations are not independent, for if two angles of the first triangle are equal to two angles of the second triangle each to each, the third angles also must be equal by Theor 16, and the third relation is thus only a consequence of the first two Hence without some further data we cannot conclude that the triangles are congruent

8 (i) See Theor 12

(ii) In the Fig of Theor 12, let OQ be an oblique such that $\angle COQ = \angle COP$

Then the complements of these angles are also equal,

that is, $\angle OQC = \angle OPC$ [*Th* 16]

$OP = OQ$ [*Th* 6]

(iii) Let OR, OP be obliques on *opposite* sides of OC, such that $\angle ROC$ is greater than $\angle POC$

Then $\angle ORC$ being the compt of $\angle ROC$ is *less* than $\angle OPC$ which is the compt of $\angle POC$

OR is greater than OP [*Th* 10]

Also if OR, OQ be obliques on the *same* side of OC, such that $\angle COR$ is greater than $\angle COQ$, make, on the opposite side of OC, an $\angle COP = \angle COQ$.

Then $OP = OQ$ by (ii), and $OR > OP$ by the above,

$OR > OQ$.

9 Let ABC, DEF be two triangles having BA, AC equal to ED, DF, each to each, and $\angle ABC = \angle DEF$

Then if $\angle BAC = \angle EDF$, the triangles are congruent by Theor 17, and $\angle ACB = \angle DFE$

But if $\angle BAC$ is not equal to the $\angle EDF$, let $\angle BAC$ be the greater, and at D draw a line DG, making with ED an angle $EDG = \angle BAC$, and cutting EF produced at G

Then the Δ^s BAC, EDG are congruent by Theor 17

$DG = AC$, and $\angle DGF = \angle ACB$

But $DF = AC$ [*Hyp*], $DG = DF$, and $\angle DGF = \angle DFG$ [*Th* 5]

Now the $\angle^s$ DFE, DFG are supplementary [*Th* 1]

the $\angle^s$ DFE, DGF are supplementary

That is, the $\angle^s$ DFE, ACB are supplementary

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1 In quad[l] ABCD, if BD is joined,

$$\text{because}\begin{cases} AB = CD, \\ AD = CB, \\ BD \text{ is common,} \end{cases}$$

$\triangle^s$ BAD, DCB are identically equal [*Th* 7]

$\angle ABD = \angle CDB$,

AB is par[l] to CD [*Th* 13]

AB and DC are equal and parallel, hence, by Theor 20, it follows that ABCD is a par[m]

2 In the quad[l] ABCD,

let $\angle DAB = \angle BCD$,

and $\angle ABC = \angle ADC$,

then the sum of two adjacent $\angle^s = \frac{1}{2}$ sum of the $\angle^s$ of fig ABCD

$= 2$ rt $\angle^s$, [*Th* 16 *Inf* 5]

the opposite sides are parallel [*Th* 13]

3 Let AC, BD be the diagonals, bisecting each other at O

Then in $\triangle^s$ AOD, COB,

$$\text{because}\begin{cases} AO = CO, \\ \text{and } OD = OB, & [\textit{Hyp}] \\ \text{and } \angle AOD = \text{vert opp } \angle COB, & [\textit{Th } 3] \end{cases}$$

$AD = CB$,

and $\angle OAD = \angle OCB$ [*Th* 4]

But these are alternate angles, AD is par[l] to BC [*Th* 13]

Hence AD, BC are equal and parallel, and the figure ABCD is a par[m] [*Th* 20]

4 Let ABCD be a rhombus, and O the intersection of its diagonals Then we have in the $\triangle^s$ AOD, COD,

$$\text{because}\begin{cases} AO = CO, & [\textit{Th } 21 \textit{ Cor } 3] \\ \text{and } DO \text{ is common,} \\ \text{and } AD = CD, & [\textit{Hyp}] \end{cases}$$

$\angle AOD = \angle COD$, [*Th* 7]

and these being adjacent angles, each is a rt $\angle$,

DB is perp to AC

5 In the quad^l ABCD, let AC = BD

Then in the $\triangle^s$ ABC, DCB,

$$\text{because}\begin{cases} AB = DC, \\ BC \text{ is common,} \\ \text{and } AC = DB, \end{cases}$$

$\angle ABC = \angle DCB$ [*Th* 7], each is a rt $\angle$ [*Th* 14]

6 Because the sum of the $\angle^s$ ABC, BCD is two rt $\angle^s$ [*Th* 14] and neither is a rt $\angle$, they are unequal

Then in the $\triangle^s$ ABC, DCB,

$$\text{because}\begin{cases} AB = DC, & [Th\ 21] \\ \text{and } BC \text{ is common,} \\ \text{but } \angle ABC \text{ is unequal to } \angle DCB, \end{cases}$$

AC, BD are unequal [*Th* 19]

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1 Let AC be a diagonal of the rhombus ABCD

Then the $\triangle^s$ ABC, ADC are congruent [*Th* 7]

$\angle CAB = \angle CAD$

Similarly $\angle ACB = \angle ACD$

Hence if the figure is folded about the diagonal AC, AD will coincide in direction with AB, and CD will coincide in direction with CB.

Thus D, the common pt of AD, DC, must coincide with B, the common pt of AB, BC

the $\triangle^s$ ABC, ADC can be made to coincide, and hence the rhombus is symmetrical about the diagonal AC and, similarly, also about BD

2 As in Ex 1 we can prove that each diagonal of a square bisects the angles through which it passes, and that therefore the triangles on either side of a diagonal coincide when the figure is folded about that diagonal

Hence a square is symmetrical about either diagonal

The other axes of symmetry are the lines joining the middle pts of opposite sides

3 The triangles though congruent are not similarly placed with regard to the diagonal For since two adjacent sides of a rectangle are unequal, they cannot coincide when the

figure is folded about the diagonal through their pt of intersection Thus this diagonal, and similarly the other also, cannot be an axis of symmetry

The only axes in this case are the lines joining the middle pts of opposite sides

4 Let ABCD be an oblique par^{m}, E, F the mid-pts of AB and CD

Then since DA, DC are unequal, they cannot coincide when the figure is folded about DB Thus the diagonals are not axes of symmetry

Again the lines AE, DF are equal, for they are halves of equal lines AB, CD, and EF is parallel to AD [*Th* 20]

Hence the ∠BEF=the int opp ∠BAD [*Th* 14], and is not a rt angle [*Hyp*]

the ∠^{s} BEF, AEF are unequal

Hence the lines EB, EA do not coincide when the figure is folded about EF

Thus the lines joining the mid-pts of opposite sides are not axes of symmetry, and no axes of symmetry exist

5 Since BA, BC are unequal, the figure is not symmetrical about BD

But in the △^{s} BAC, DAC,

$$\text{because}\begin{cases} AB=AD, & [Hyp] \\ \text{and } CB=CD, & \\ \text{and } AC \text{ is common}, & \end{cases}$$

∠BAC=∠DAC, and ∠BCA=∠DCA [*Th* 7]

Thus these triangles coincide when the figure is folded about AC, and AC is therefore an axis of symmetry

6 (i) Let ABCD, EFGH be par^{ms} in which AB=EF, BC=FG, and ∠ABC=∠EFG Join AC, EG

Then the △ABC can be made to coincide with the △EFG [*Th* 4], and the △ADC will then coincide with the △EHG [*Th* 7]

Hence the two par^{ms} coincide, and are therefore identically equal

(ii) Let ABCD, EFGH be two rectangles, in which AB=EF and BC=FG

Then the ∠ABC=the ∠EFG, being both rt ∠^{s}

Hence the example is a particular case of (i)

7 Apply the $\triangle$ BAD to the $\triangle$ FEH so that A falls on E and AB along EF Then, since $\angle$BAD$=\angle$FEH, the line AD will fall along EH, and, because AB$=$EF and AD$=$EH, the pts B, D will fall on F and H Also the $\triangle^s$ BDC, FHG are congruent by Theor 7, hence $\angle$BDC$=\angle$FHG, and $\angle$DBC$=\angle$HFG Thus BC, DC fall on FG, HG respectively and C coincides with G Thus the quadrilaterals coincide.

8. Let ABCD be the par^m, O the middle pt of the diag BD, and POQ a st line through O terminated by the sides APB, CQD

Then in $\triangle^s$ POB, QOD,

because $\begin{cases} \angle\text{POB}=\text{vert opp } \angle\text{QOD}, & [Th\ 3] \\ \text{and } \angle\text{OBP}=\text{alt } \angle\text{ODQ}, & [Th\ 14] \\ \text{and BO}=\text{OD}, & [Hyp] \end{cases}$

PO$=$OQ [*Th* 17]

9 Let AP, CQ be perp to diag BD Then

because $\begin{cases} \angle\text{ADP}=\text{alt } \angle\text{CBQ}, & [Th\ 14] \\ \text{and } \angle\text{APD}=\angle\text{CQB, being rt } \angle^s, & \\ \text{and AD}=\text{CB}, & [Th\ 21] \end{cases}$

$\triangle^s$ APD, CQB are identically equal, [*Th* 17]

AP$=$CQ

10. AX and YC are equal, being halves of the equal sides AD and BC They are also parallel Hence AY is par^l to CX, and AYCX is a par^m [*Th* 20]

11 Join BE, AD, CF

Then, because AB is equal and parallel to DE,

AD is equal and par^l to BE [*Th* 20]

Similarly CF is equal and par^l to BE

AD and CF are both equal and parallel [*Th* 15]

AC, DF are both equal and parallel [*Th* 20]

12. In the Fig considered, CD is greater than AB

From A, B draw AP, BQ perp to CD Then APQB is a par^m, and AP$=$BQ

Then in the *rt angled* $\triangle^s$ APD, BQC,

because hypot AD$=$hypot BC, and AP$=$BQ,

$\angle$ADP$=\angle$BCQ, and PD$=$CQ, [*Th* 18]

(i) Hence $\angle A+\angle C=\angle BAD+\angle ADP=180°$ [*Th* 14]
Similarly $\angle B+\angle D=180°$
$\angle A+\angle C=\angle B+\angle D$

(ii) Adding PQ to the equal lengths PD, CQ, we have QD = PC
Hence in the Δ^s BQD, APC,

$$\text{because}\left\{\begin{array}{l} BQ=AP, \\ \text{and } QD=PC, \\ \text{and } \angle BQD=\angle APC, \text{ being rt } \angle^s, \end{array}\right.$$

BD = AC [*Th* 4]

(iii) Also, if X is the mid-pt of AB, and Y the mid-pt of PQ, and also of CD, then it can be shewn by Theor 20 that XY is par[l] to AP and in consequence perp to CD and AB

Moreover XA = XB, and YD = YC

Hence if the figure be folded over about the line XY, the lines XA, YD will fall along XB and YC and be terminated at B and C, that is the figure is symmetrical about XY

13 Since the lines are initially parallel, the alternate angles BAP, ABQ are equal by Theor 14

Let AP′, BQ′ be the new positions of AP, BQ at the end of any time

(i) Then, since the rods are turning at equal rates, the angles through which they have turned, viz. $\angle^s$ PAP′, QBQ′, are equal

Hence, by subtraction, the $\angle^s$ BAP′, ABQ′ are equal

And these are alternate angles, AP′ is par[l] to BQ′

Thus the lines are always parallel

(ii) Join P′Q′ cutting AB in C

Then in the Δ^s ACP′, BCQ′

$$\text{because}\left\{\begin{array}{ll} \angle ACP=\text{vert opp } \angle BCQ', & [\textit{Th } 3] \\ \angle CAP'=\angle CBQ', & [\textit{Proved}] \\ \text{and } AP'=BQ', & [\textit{Hyp}] \end{array}\right.$$

AC = BC [*Th* 17]

Hence C is the mid-pt of AB and is therefore a fixed pt

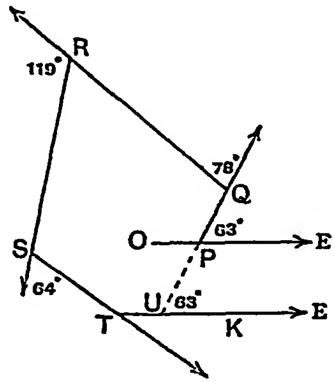
Thus the line joining the ends of the rods always passes through a fixed pt

14 We have $A=\frac{3}{7}(180^\circ-A)$, whence $A=54^\circ$.

$$B+C=126^\circ \qquad [Th\ 16]$$

But $B=\frac{4}{3}C$; $\quad \frac{7}{3}C=126^\circ$, whence $C=54^\circ$, and $\quad B=72^\circ$

15. In the diagram P, Q, R, S, T represent the pts at which the yacht changes her course, and TK her final direction QP is produced to meet TK at U Then since TK is par[1] to OP,



$$\angle PUK=\angle QPE=63^\circ$$

Now QRSTU forms a pentagon and the sum of its ext. angles $=360^\circ$ [*Th* 16 *Cor* 2]

But the ext. angles at Q, R, S, U $=324^\circ$,

$$\text{ext angle at } T=360^\circ-324^\circ=36^\circ$$

Thus the yacht must change her course by 36°

16 The sum of the ext angles of the figure = 4 rt ∠[s] [*Th* 16 *Cor* 2]

sum of the int angles is also 4 rt ∠[s] [*Hyp*]

Hence the sum of the ext and int angles = 8 rt ∠[s]

But the ext and int angles at any vertex = 2 rt ∠[s]

the no of vertices, and of sides $=8\div2=4$

17 The sum of the int angles of the five-sided figure $=2\times5-4$, or 6 rt ∠[s] [*Th.* 16 *Cor* 1]

$$\angle B+\angle C+\angle D+\angle E=470^\circ$$

$$\angle A=540^\circ-470^\circ=70^\circ$$

$$\angle EAB+\angle ABC=180^\circ$$

AE and BC are parallel [*Th* 13]

18. (i) The lines AP, BQ are par[1] when $\angle PAB+\angle QBA=180^\circ$ [*Th* 13]

Now the sum of ∠[s] PAB, QBA is increased by $7\frac{1}{2}^\circ+3\frac{3}{4}^\circ$, that is $11\frac{1}{4}^\circ$, every second

$$\text{time before AP, BQ are par}^{l}=\frac{180}{11\frac{1}{4}}\text{ sec}=16\text{ seconds}$$

(ii) Angle between AP and BQ = suppt of the sum of $\angle^s$ PAB, QBA, [*Th* 16]

$$= 180° - 11\tfrac{1}{4}° \times 12,$$
$$= 45°$$

(iii) The angle between AP and BQ diminishes each second by the amount by which the sum of $\angle^s$ BAP, QBA is increased, for the sum of the 3 angles is constant

Hence the rate of decrease is $11\tfrac{1}{4}°$ per second

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3 With the Fig and construction of Ex 2, we can prove the $\triangle^s$ AYZ, CYV congruent by Theor 4

ZY = VY, CV = AZ, and $\angle$ YCV = $\angle$ YAZ,

and these being alt angles, CV is par[l] to AB [*Th* 13]

But AZ = ZB [*Hyp*] CV = BZ

Thus CV is both equal and parallel to BZ,

CVZB is a par[m] [*Th* 20]

BC = ZV = 2 ZY

4 In the Fig of Ex 1 let X, Y, Z be the mid-pts of the sides

Then, by Ex 2, ZY is par[1] to BX, and XY is par[1] to BZ

Thus BZYX is a par[m], and XZ being a diagonal, the $\triangle^s$ BZX, YXZ are identically equal [*Th* 21]

Similarly the $\triangle^s$ AZY, YXC are identically equal to $\triangle$ XYZ

all four triangles are identically equal

5 In the Fig of Ex 2 let ADE be a st line cutting ZY in D and BC in E

Then ZY is par[l] to BC [*Ex* 2]

, in $\triangle$ ABE, the line ZY drawn through the mid-pt Z of AB, parallel to the base BE, bisects the third side AE [*Ex* 1]

6. Let AC meet BX in E and DY in F

Then BY and XD are par[l], and also equal, being halves of equal sides

BX is par[1] to YD [*Th* 20]

Then from $\triangle$ BEC, because YF is par[1] to BE, CF = EF [*Ex* 1]

And in $\triangle$ AFD, because XE is par[1] to DF, AE = EF [*Ex* 1]

AC is trisected at E, F

7. In the quad^l ABCD, let P, Q, R, S be middle pts of sides AB, BC, CD, DA respectively Then by Ex 2, PQ and SR are each par^l to AC, and PS and QR are each par^l to BD

8 In last Ex, PR and QS are diags of a par^m, and therefore bisect each other [*Th* 21 *Cor* 3]

9 Through A draw ALM, par^l to CD, cutting OX, BQ (produced if necessary) in L and M

Then AP, OX, BQ are par^l, for they are all perp to CD

PXLA, LXQM are par^{ms}

$AP = LX = MQ$, and $LX = \frac{1}{2}(AP + MQ)$

Also by Theor 22 Cor, or Ex 3, because O is the mid-pt of side AB of $\triangle$ ABM and OL is drawn par^l to side BM,

$OL = \frac{1}{2}BM$

in case (i), by addition,

$OX = OL + LX = \frac{1}{2}(BM + AP + MQ) = \frac{1}{2}(AP + BQ)$,

and in case (ii), by subtraction,

$OX = OL \sim LX = \frac{1}{2}\{BM \sim (MQ + AP)\} = \frac{1}{2}(BQ \sim AP)$

Thus in the numerical Ex,

$OX = \frac{1}{2}(5\ 8 + 4\ 2)$cm $= 5$ cm, if A, B are on the *same* side of CD

$OX = \frac{1}{2}(5\ 8 - 4\ 2)$cm $= 0\ 8$ cm, if A, B are on *opposite* sides of CD

10. With the Fig of Theor 22, where the three parallels AB, CD, EF cut off equal intercepts XY, YZ and PQ, QR respectively from the two transversals XZ, PR, we have

because PXMQ, QYNR are par^{ms},

$PX = QM$ and $QY = RN$

Also $MY = NZ$, [*Th* 22]

$PX + RZ = QM + QY + NZ$

$= QM + QY + MY$

$= 2QY$.

That is, QY is the Arith Mean between PX, RZ

11. Let P, Q be the middle pts of AB, CD, the slant sides of the trapezium ABCD

Then, by Theor 22, the line drawn through P par^l to AD must bisect CD, *i e* passes through Q, hence PQ is par^l to AD

by Ex 10, PQ is the Arith Mean between *a* and *b* and its length is consequently $\frac{1}{2}(a+b)$cm

• 12 Let $1a$, $2b$, $3c$, $4d$, $5e$ be the five parallels

Through 1, 2, 4, 5 draw 1P, 2Q, 4R, 5S par^{l} to OY and meeting $3c$, produced if necessary, in P, Q, R, S respectively

Then by Theor 17, the $\triangle^{s}$ 13P, 53S are congruent $3S=3P$

$$1a+5e=(3c-3P)+(3c+3S)$$
$$=\text{twice } 3c$$

Similarly $2b+4d=\text{twice } 3c$

the five lines together $=$ five times $3c$, that is, $3c$ is the mean of the five parallels

Similarly if $(2n+1)$ parallels be so drawn the $(n+1)^{\text{th}}$ parallel is the mean of all the $(2n+1)$

13 Let ABCD be the given par^{m}, and let AX, CY be perps on the given line from one pair of opp $\angle^{s}$, and DP, BQ perps from the other pair of opp $\angle^{s}$ Let the diagonals intersect in E, and let EF be perp to the given line

Then $AX+CY=2EF$ [*Ex* 9 p 65]

$=DP+BQ$,

since E is the middle pt of the diagonals [*Th* 21 *Cor* 3]

14 From D in the base BC let DE, DF be drawn perp to AC, AB respectively, from B let BG be drawn perp to AC

Draw BH par^{l} to AC to meet ED produced in H

Then GH is a par^{m} by construction, and $EH=BG$

Now because BH is par^{l} to AC, $\angle HBD=\text{alt } \angle ACB=\angle ABC$ [*Th* 14 and 5]

Then in the $\triangle^{s}$ BDF, BDH,

$$\text{because}\begin{cases}\angle DFB=\angle DHB, \text{ being rt } \angle^{s}, \\ \text{and } \angle DBF=\angle DBH, \quad [\textit{Proved}] \\ \text{and } DB \text{ is common},\end{cases}$$

$DF=DH$ [*Th* 17]

$DE+DF=ED+DH=EH=BG$

If D lie on CB produced, we have $DE \sim DF=BG$

15 Let OX, OY, OZ be perps to BC, CA, AB respectively

Through O draw POQ par^{l} to BC

Through A draw AGH perp to PQ and BC

Then $\angle^{s}$ APQ, AQP are respectively equal to $\angle^{s}$ ABC, ACB,

$\triangle$ APQ is equiangular to $\triangle$ ABC, and equilateral, [*Th* 6 *Cor*]

∴ by the last Ex, OY + OZ = perp from P on AQ,
= perp from A on PQ, by congruent triangles,
= AG
∴ OX + OY + OZ = AG + OX
= AG + GH (for OGHX is a par^{m})
= AH, which is independent of the position of O,
and ∴ constant

16 Let AB, CD be two equal and par^{l} st lines, and XY, PQ their projections on any st line

Through A, C draw AE, CF par^{l} to XY to meet BY, DQ in E, F respectively

Then AEYX, CFQP are par^{ms}, ∴ AE = XY and CF = PQ

But in the $\triangle^{s}$ AEB, CFD,

because BA, AE are parallel to DC, CF respectively,

$\left\{\begin{array}{l} \angle BAE = \angle DCF, \\ \text{and } \angle AEB = \angle CFD, \text{ being rt } \angle^{s}, \\ \text{and } AB = CD, \end{array}\right.$ [*Ex* 4, p 41] [*Hyp*]

∴ AE = CF [*Th* 17]

That is, XY = PQ

Page 68.

1. Use the diagonal scale of inches as explained on pp 66, 67

3 Measuring the distance 5·7 cm on the dividers and transferring to a diagonal scale of inches, it is found to be equivalent to 2·24″

By calculation, 5·7 cm = 0·3937″ × 5·7
= 2·244″, to 3 dec places

4 By measurement, 3·15 inches = 8·0 cm,
∴ whence 1 cm = 3·15″ ÷ 8
= 0·39″, to 2 dec. places

5 By measurement, 2·9 cm = 1·14″,
whence 1″ = 2·9 cm ÷ 1·14 = 2·543 cm

Also by measurement, 6·2 cm = 2·44″,
whence 1″ = 6·2 cm ÷ 2·44 = 2·541 cm

Thus the average result is 1″ = 2·54 cm, to 2 dec places

6 The req[d] lines will be 3.36″ and 4.08″ Measure these lengths by use of a diagonal scale of inches

7 The req[d] lines will be 0.85″, 2.98″, 1.01″, which must be measured by means of a diagonal scale

8 The length of the line $= 4.17''$

$= 10.6$ cm (by measurement)

9 Since 5 km are represented by 1 cm

42.5 km   8.5 cm

and, by measurement, 8.5 cm $= 3.35''$

10 $2.75''$ represent 55 mi

$1''$ represents 55 mi $\div 2.75$, *i.e.* 20 mi

But 1 cm $= 0.3937''$, 1 km $= \frac{5}{8}$ mi nearly

$0.3937''$ represent $(20 \times 0.3937 \times \frac{8}{5})$ km,

that is, 1 cm represents 12.6 km (approx.)

11 $4.2''$ represents (35×4.2) mi, *i.e.* 147 miles

distance $= (147 \times \frac{8}{5})$ km $= 235$ km (to nearest km)

Again, by measurement, $4.2'' = 10.7$ cm

10.7 cm represent 235 km

or 1 cm represents $\dfrac{235}{10.7}$ km, *i.e.* 22 km (to nearest km)

12 Scale on first map is $2\frac{1}{2}''$ to $37\frac{1}{2}$ mi

i.e. $1''$ to $(37\frac{1}{2} \div 2\frac{1}{2})$ mi

i.e. 1 inch to 15 miles

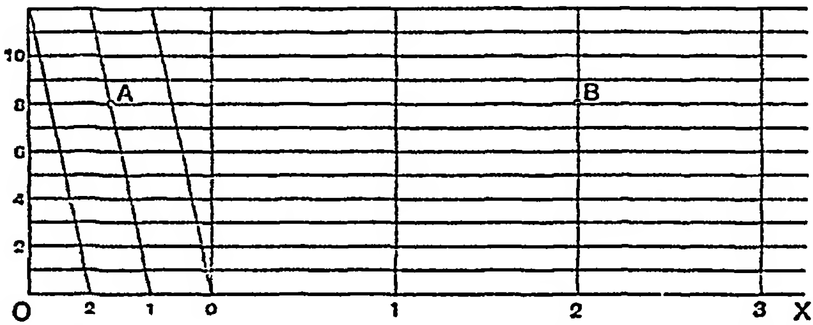
Scale on second map is 7 cm to 88 km

i.e. $7 \times 0.3937''$ to $\left(88 \times \dfrac{5}{8}\right)$ mi, nearly

i.e. $1''$ to $\dfrac{55}{2.7559}$ mi, nearly

i.e. 1 inch to 20 miles, nearly

13



Draw a line OX and set off 4 or 5 divisions, each 2 cm long, to represent yards

At O draw OY perp to OX and of any convenient length

By means of Prob 7 divide the first division of OX into 3 equal parts and OY into 12 equal parts

Through the pts of division on OY draw parallels to OX with the set squares, and complete the figure as in the diagram

Then a line such as AB represents 2 yds. 1 ft 8 in

Page 79.

1 Describe an equil $\triangle$ ABC, by taking any line BC and with centres B, C, and radius BC, drawing arcs to cut at A

Then the three angles are equal [*Th* 5], and their sum is 180° [*Th* 16], each is 60°

Now bisect $\angle$ABC by Prob 1, and then bisect each of the two parts

2 (i) Let ABC be a rt $\angle$ Cut off BC any convenient length on one of the arms, and on BC describe an equil $\triangle$ DBC

Then $\angle$ABD $= 90° - 60° = 30°$

Bisect $\angle$DBC by BK, then $\angle$DBK $=$ $\angle$KBC $= 30°$

Thus BD, BK trisect the rt $\angle$ABC

(ii) Let PQR be an angle of 45° Draw QS making with QR an angle of 60°, on the same side as QP

Bisect $\angle$SQR by QX, and bisect $\angle$XQR by QY

Then $\angle$PQR is trisected, for each of its three parts is 15°

3 Use the construction of Prob 7

Length $= \frac{1}{5}$ of 6 7 cm $= \frac{1}{5} \times 6\ 7 \times 0\ 3937''$

$= 0\ 53'$, to the nearest hundredth

4 Use Prob 7

Length $= \frac{1}{7}$ of $3\ 72''$

$= \dfrac{3\ 72}{7 \times 0\ 3937}$ cm

$= 1\ 319$ cm

$= 1\ 3$ cm, to the nearest mm

6. Bisect AB at C Through C draw CD perp to AB to meet XY in D

Join DA, DB

Then the Δ^s ACD, BCD are congruent by Theor 4

DA, DB are equal, and D lies in XY

The construction is impossible when XY is parallel to CD, *i e* when XY is perp to AB

7 Bisect the $\angle$BAC by AD cutting XY in D Through D draw DP, DQ perp to AB, AC

Then the Δ^s DAP, DAQ are congruent by Theor 17

DP = DQ

D, a pt on XY, is equidistant from AB, AC

The construction is impossible if AD, XY are parallel, *i e* if XY is parallel to the internal bisector of $\angle$BAC

8 At C, any pt in AB, make the $\angle$BCD of the given magnitude

Through P draw PQ par[l] to CD, and cutting AB in Q

Then $\angle$PQB = $\angle$DCB, by Theor 14

9 By Theor 4 the Δ^s PHK, P'HK are congruent.

$\angle$PKH = $\angle$P'KH But $\angle$P'KH = vert opp $\angle$QKB

$\angle$PKH = $\angle$QKB

10 (i) Through P draw PHK par[l] to AB Then if AH, BK are the perps from A, B on PK, the fig AHKB is a par[m], and AH = BK

Or (ii) bisect AB at Q Join PQ, and from A, B draw perps AC, BD, to PQ or PQ produced

Then the Δ^s ACQ, BDQ are congruent [*Th* 3 and 17]

AC = BD

Both constructions fail if P, A, B are in the same st line

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(i) one (ii) one (iii) one (iv) none

Page 84

2 By measurement, BX=1.57", XC=1.43"

$$\text{Hence } \frac{BX}{XC}=\frac{1.57}{1.43}=1.098,$$

$$\text{and } \frac{c}{b}=\frac{2.75}{2.5}=1.10,$$

which, allowing for probable errors in measurement, may be regarded as identical

3 Draw BC equal to 3.15" to represent 315 yds At B make an angle CBA equal to 39° and cut off BA=2.6" to represent 260 yds Then AC is found by measurement to be 2.00", which represents 200 yds

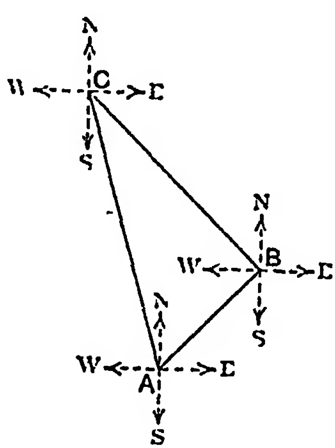
4 $$\angle A=180°-(47°+68°)=65°$$

Draw BC equal to 7.5 cm At B, C make angles CBA, BCA equal to 47°, 68° respectively

By measurement in the plan, AB=7.7 cm, AC=6.1 cm, and the perp from A on BC=5.6 cm

Hence the corresponding lengths in the field are 77 m, 61 m, 56 m

5 The diagram is drawn to half the given scale, A, B, C being the three turning pts of the course



By calculation AB=3 knots, BC=5¼ knots Set off AB equal to 6 cm in direction N E At B set off BC equal to 10½ cm in direction N W Then by measurement CA=12.1 cm, and ∠SCA=15°, nearly Hence the distance from the harbour =6.05 knots, and the new course is 15° E of South

6 See Prob 10

$\sqrt{c^2-a^2}=\sqrt{(10\ 6)^2-(5\ 6)^2}=\sqrt{81}=9$, which should be the number of centimetres in b as obtained by measurement (See Theor 29)

7 See Prob 9

The two values of C must be supplementary, for, in Prob 9, $AC_1=AC_2$, and $\angle AC_1C_2=\angle AC_2C_1$ But $\angle$s AC_2B, AC_2C_1 are supplementary, $\angle$s AC_2B, AC_1B are supplementary

8. Draw a line AX, at A make $\angle XAY$ equal to 50°, from AY cut off AC equal to 6 5 cm Then with centre C and the given values of a as radii in turn, draw circles and note the pts where these meet AX

In (i) there are 2 pts of intersection, but on different sides of A, so that one $\triangle$ has A=50° and the other has A=130° Hence there is only *one* solution

In (ii) there are 2 pts of intersection on same side of A Hence each possible $\triangle$ has A=50°, and there are *two* solutions

In (iii) the 2 pts of intersection coincide, as the O touches AX, hence *one* solution

In (iv) the O does not cut AX at all, and there is *no* solution

9 It will be seen that we have to draw a triangle rt-angled at A and with AB, BC given lengths Hence use Prob 10 A convenient scale is 1 inch to 100 yds It will then be found, by measurement, that AC=3 80′, thus the required distance is 380 yds

10 Draw the base AB, 4 cm in length Bisect AB at C Draw CD perp to AB and equal to 6 2 cm Then $\triangle$ ADB is isosceles, for by Theor 4, the $\triangle$s ACD, BCD are congruent, and therefore DA=DB

11 Let the magnitude of the given angle be denoted by X

Draw AD of the given length representing the perp from the vertex A on the base Through D draw BDC perp to AD From A draw AB, AC on opp sides of AD, making each of the $\angle$s DAB, DAC equal to $\frac{X}{2}$ Then the $\triangle$ ABC is isos, for the $\triangle$s ADB, ADC are congruent by Theor 17, and the vert $\angle BAC=X$

Draw AD of length 6 cm, and proceed as above making each of the angles DAB, DAC equal to 30° Then each angle of the $\triangle$ is 60°

12. Draw AD of length 5 0 cm to represent the perp from vertex A on the base Through D draw BDC perp to AD With A as centre draw one arc with radius 5 8 cm. cutting BC in B_1, B_2, and a second arc with radius 9 0 cm cutting BC in C_1, C_2

Then each of the $\triangle^s$ AB_1C_1, AB_1C_2, AB_2C_1, AB_2C_2 satisfies the data, but it will be seen that they may be grouped in pairs, each of one pair having its base 4 5 cm while that of each of the others is 10 4 cm

13 Draw two par[1] lines BC, EF whose distance apart = P

At any pt A in EF make $\angle$ EAB = L, and $\angle$ FAC = M [*Prob* 5] Then $\angle$ ABC = $\angle$ EAB, and $\angle$ ACB = $\angle$ FAC [*Th* 14]

ABC is the required $\triangle$

14. Draw AC equal to b At C make $\angle$ ACK equal to $\angle$ C At any pt K in CK make $\angle$ CKF on the same side of CK as A and equal to $\angle$ B [*Prob* 5] Through A draw AB par[1] to FK and meeting CK in B.

Then $\angle$ ABC = $\angle$ CKF [*Th* 14], and $\angle$ CKF was made equal to the given $\angle$ B

ABC is the required $\triangle$

15. Draw BC equal to the given base At B make $\angle$ CBA equal to $90° - \frac{1}{2}L$ At C make $\angle$ BCA also equal to $90° - \frac{1}{2}L$ Then $\triangle$ ABC is isosceles [*Th* 6], and its vertical angle

$$= 180° - (B + C) = L$$

16. Draw PB equal to the given length $(a+b)$ At P draw PR making $\angle$ BPR = 45° With centre B and radius equal to c describe a circle cutting PR in A, A′ From A, A′ draw perps AC, A′C′ to PB Then either of the $\triangle^s$ ACB, A′C′B will satisfy the given conditions

For $\angle$ PAC = 45° [*Th* 16], and $\angle$ APC = $\angle$ PAC, whence CA = CP

AC + CB = PB, and similarly A′C′ + C′B = PB

NOTE There will be two, one, or no solutions according as c is greater than, equal to, or less than the perp from B to PR

In the given case there are two solutions, but, as could be proved in the general case, the $\triangle^s$ are congruent, and the lengths of the sides are 4 5 cm and 2 8 cm

$$\sqrt{a^2+b^2} = \sqrt{(4\ 5)^2+(2\ 8)^2} = \sqrt{28\ 09} = 5\ 3 \text{ cm}$$

17 Let B, C be the given base angles, PQ the given perimeter

From P draw PA making $\angle QPA = \frac{1}{2}B$

From Q draw QA making $\angle PQA = \frac{1}{2}C$

From A draw AB making $\angle PAB = \angle QPA$, to cut PQ in B

From A draw AC making $\angle QAC = \angle PQA$, to cut PQ in C

Then $BA = BP$ [*Th* 6], ext $\angle ABC = \angle BPA + \angle PAB = B$

Similarly $CA = CQ$, and $\angle ACB = C$

∴ ABC is the required △

18 Draw an $\angle PBC = 60°$ Cut off $BC = 6.5$ cm, and $BP = 10$ cm

Join PC, and from C draw CA cutting BP in A and making

$\angle PCA = \angle BPC$

Then $AP = AC$ [*Th* 6], and $BA + AC = BP$

Hence ABC is the required △

19 Draw $\angle PBC = 55°$ Cut off $BC = 7$ cm, and $BP = 1$ cm

Join PC, and from C draw CA cutting BP produced in A, and making $\angle PCA = \angle APC$

Then $AC = AP$ [*Th* 6], and $BA - AC = BP$

Hence ABC is the required △

By measurement $AB = 8$ cm, $AC = 7$ cm

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1 On PQ as base, and on opp sides of it construct two equil △s PRQ, PSQ Then the four sides of the figure PRQS are all equal to the diagonal PQ. Also each angle of an equil △ is 60°, $\angle SQP = \angle RPQ$, whence PR is par[l] to QS Similarly RQ is par[l] to PS Hence the figure is a rhombus

Each of the ∠s R, S = an angle of an equil △ = 60°

Each of the ∠s Q, P = sum of two angles of an equil. △ = 120°

2 See p 19, Ex 3

3 Let AB be the given diagonal Bisect AB at O Draw COD perp to AB, making $OC = OD = OA$ Then CADB is the required square

The four △s formed can be proved congruent by Theor 4, hence the four lines AC, CB, BD, DA are all equal, and the figure is equilateral

In △AOC, $\angle ACO = \angle OAC$ [*Th* 5] and their sum is 90° [*Th* 16],

∴ each is 45°. Hence ∠ACO=45°; similarly ∠BCO=45° ∴ ∠ACB=90°. Similarly each of the other ∠s of the figure is a rt. ∠. Thus the figure is both equilateral and rectangular, and is ∴ a square.

4. Construct a △AOB having AB=5.5 cm., AO=4 cm., OB=3 cm. Produce AO, BO to C and D so that OC=AO and OD=OB. Then ABCD will be the required par^m.

By Theor. 4 the △s AOD, COB are congruent.

∴ ∠ODA=∠OBC; whence AD is par^l to BC. [Th. 13.]

Similarly DC is par^l to AB. By measurement AD=4.4 cm.

5. The five independent data are the four segments of the two diagonals and the angle at which the lines cut.

Let AOC, BOD be the diagonals and the ∠AOD=60°. Then by Theors. 4 and 13 we can prove AD par^l to BC and AB par^l to CD. Hence the figure is a par^m.

By measurement AD=3.0 cm., AB=5.2 cm. Thus the perimeter=2(AD+AB)=16.4 cm.

In the second case, by measurement each side=4.24 cm., thus the perimeter=16.97 cm.

$$\text{increase per cent} = \frac{0.57}{16.4} \times 100 = 3.4 \text{ nearly.}$$

6. By assuming different magnitudes for the angle BAD and proceeding as in Prob. 11, we should get several quadrilaterals differing in shape but each satisfying the given conditions. Hence the shape is not completely determined by the data.

If A=100°, it will be found that the length of DB is greater than 6.5 cm. Hence circles with centres D, B and radii 4.0 cm. and 2.5 cm. respectively do not intersect.

If BD is *equal to* 6.5 cm., the two circles will meet at a point C on the line BD. For any value of BD *greater than* 6.5 cm. the construction fails: thus when BD is least the fig. reduces to a triangle BAD with AD=3.3 cm., AB=5.6 cm., DB=6.5 cm., whence we find by measurement that A=90°.

7. Let BD be the given diagonal. First construct a △ with sides equal to DA, AB, BD, and then on the other side of BD draw a △ whose remaining sides are equal to DC, CB.

By Theor. 11 any side of a △ must be less than the sum of the other two sides, and also greater than their difference. Thus BD must be greater than $a \sim d$ and $b \sim c$, and less than $a+d$ and $b+c$.

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1 The locus is a concentric circle, whose radius is equal to the sum or difference of the radius of the given circle and the given distance

2 Join AB and bisect it at C, draw CD perp to AB meeting RQ in P Then this is the req^d pt, for every pt on CD is equidistant from A and B [*Prob* 14]

3 All pts which are equidistant from A and B lie on the st line which bisects AB at rt ∠^{s} The intersections of this line and the circumference give two pts, each of which satisfies the given conditions

4. Produce AB and CD to meet in O Draw OX, OY the internal and external bisectors of ∠BOD By Prob 15, any pt on either of these lines is equidistant from AB and CD The intersections of OX, OY with RQ therefore give two positions of the pt P

5 All pts 4 cm distant from A lie on a ○, centre A, radius 4 cm
All pts 5 cm distant from B lie on a ○, centre B, radius 5 cm
The common pts of the two circles are the pts required.

6 All pts 3 cm distant from AB lie on one of the two lines EF, GH which are par^l to AB on either side of it and at a distance of 3 cm from it (See p 90 *Ex* 2)

Similarly all pts 4 cm distant from CD lie on either PQ or RS which are par^l to CD and 4 cm from it on either side

The 4 intersections of EF, GH with PQ, RS give 4 solutions

7 Let two perp st lines OX, OY represent the rulers, let AB be any position of the moving rod, and C the mid-pt of AB

From Prob 10 we see that the line joining the vertex of a rt angled △ to the mid-pt of the hypotenuse = half the hypotenuse

$$OC = \tfrac{1}{2}AB = \text{a constant}$$

C always lies on a circle with the fixed point O as centre, and radius equal to this constant

It is obvious that C always lies within ∠XOY its locus is that quadrant of this circle which lies within ∠XOY

8 Let ACB be a rt angled Δ on AB as hypotenuse Then, if AB is bisected at D, CD is constant, being equal to half of AB [*Ex* 10, p 47]

Thus the locus of C is a circle whose centre is D and radius equal to half the given base

9. Take several positions of X along BC Bisecting each of the lines AX so obtained we conjecture from the figure that the required locus is a st line par[1] to BC

Draw AD perp to BC and bisect it at O Let X be *any* point on BC, and P the mid-pt of AX

Then, by Ex 2, p 64, OP is par[1] to DX, *i e* to BC

Similarly any other position of the pt P lies on the st line through O par[1] to BC, which is therefore the required locus

10. Let O be the centre of the given ○, C the mid-pt of AO

Take X *any* pt on the circle, and P the mid-pt of AX

Then, by Ex 3, p 64, CP=$\frac{1}{2}$OX=a constant, for OX is the radius of the given ○ Thus P is always a constant distance from the fixed pt C, and its locus is a circle with centre C and radius equal to half the radius of the given circle

11 Bisect AB in C, and AX in D Then DC is par[1] to BX [*Ex* 2, p 64] and is perp to AD Thus the locus required is that of the vertices of the rt angled Δs which can be drawn on AC as hypotenuse Now use Ex 8

12. To find a position of P, measure off along OX a length OM less than 6 cm At M draw MP perp to OX and equal to the difference between 6 cm and OM Then

$$PM+PN=PM+OM=6 \text{ cm}$$

(i) Along OX cut off OA=6 cm Join PA Then MP=MA, $\angle$MPA=$\angle$MAP=45°, for $\angle$PMA=90° Thus A being a fixed pt, and $\angle$OAP a constant $\angle$, the line AP is known in position, and the locus of P is a line through A such that $\angle$OAP=45° This line makes equal intercepts, viz 6 cm, on OX and OY

(ii) Let P be a pt such that PM−PN=3 cm Along OY cut off OB=3 cm Then, since ON−NP=3 cm, we have NP=NB, whence $\angle$YBP=45° Thus the locus of P is a st line through B, such that $\angle$YBP=45°

13 (i) From OX cut off any length OM From M draw MP perp to XO and equal to 2OM Then P is a position of the moving pt, for PM=2OM=2PN By plotting several

positions we see that the locus is a st line through O Similarly for (ii)

14 The pt must lie on the st line EF which is par[1] to the two given par[1] lines and half way between them It must also lie on the ○ whose centre is the given pt O and whose radius is the given length a The intersections of EF and this circle give possible positions of the pt

There will be two, one, or no solutions according as a is greater than, equal to, or less than, the perp from O on EF

15 All pts 2¾″ distant from MX lie on one or other of the two lines, par[1] to MX, on either side of it, and 2¾″ distant from it

All pts 2¾″ distant from S lie on a circle, centre S and radius 2¾″ It will be found that only the line on the same side of MX as S cuts the ○, the two intersections of this line with the ○ give the required pts

16 Assume the distance of S from MX to be 2″ Draw a series of lines all par[1] to MX, on the same side of it as S, and distant from it 1″, 1·2″, 1·4″, Also with centre S and radii equal to these distances describe arcs and mark the pts where each arc cuts the *corresponding* line These pts are all equidistant from S and MX

The curve formed by joining these pts is called a **parabola**

17 Let AB be the given base, h the given altitude, XY the given st line

The perp distance of vertex C from AB is h C must lie on one of the two lines par[1] to AB and at a distance h from it on either side It must also lie on XY the intersections of the two parallels with XY give two positions of C and there are two △[s] which satisfy the given conditions

18 Let ABC be given △ Bisect the ∠[s] B, C by BI, CI meeting in I Then all pts equidistant from BA, BC lie on BI, [*Prob* 15], and all pts equidistant from CA, CB lie on CI Thus I is equidistant from each of the sides

19 (i) Let P be the mid-pt of QR Draw PM, PN perp to OX, OY respectively Then PM$=\frac{1}{2}$OR, and PN$=\frac{1}{2}$OQ [*Ex* 3, p 64]

PM+PN$=\frac{1}{2}$[OR+OQ]=*constant* [*Hyp*]

locus of P is a st line cutting off from OX, OY intercepts equal to this constant [See Ex 12] Similarly for (ii)

20 (i) From S draw SQ in *any* direction making SQ 3.5" in length. Join S'Q.

From S' draw S'P making ∠QS'P equal to ∠S'QS, on same side of QS', and cutting SQ at P.

Then PS = PQ [*Th.* 5]. So that

$$SP + PS' = SP + PQ = SQ = 3.5$$

Thus any number of points such as P may be found on the required locus, which is known as an **ellipse**.

(ii) From S draw SQ, 1.5 in length, in *any* direction. Join S'Q.

From S' draw S'P, cutting SQ *produced* at P, and making ∠QS'P equal to ∠SQP.

Then P is a point on the required locus which is known as a **hyperbola**.

The proof follows the method of (i).

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1. Through A draw PAQ par[l] to BC. At A draw AB, AC making angles PAB, QAC each = X. Then, by Theor. 14, each of the lines AB, AC makes the required angle with BC.

2 Draw any line par[l] to OA and also a second line par[l] to OB and the same distance from it as the first line is from OA. The intersections of these lines give a pt. P equidistant from OA, OB.

Similarly find another pt. Q equidistant from OA, OB. Then PQ is the bisector of ∠AOB [*Prob.* 15.]

3 Produce OP to S so that PS = PO. Through S draw SQ par[l] to OB to meet OA in Q, and SR par[l] to OA to meet OB in R.

Then OQSR is a par[m] ∴ QR, OS bisect each other. [*Th.* 21 *Cor.* 3.]

∴ QR passes through P, and is bisected at that pt.

4. Take any pt. P in OB. Through P draw PQ par[l] to OC and meeting OA in Q. Draw PR par[l] to OA meeting OC in R. Then OQPR is a par[m] ∴ its diagonals bisect each other [*Th.* 21 *Cor.* 3], *i.e.* QR is bisected by OB.

Since P may be taken anywhere on OB the number of solutions is unlimited.

5 Let PQ, RS be the two par^ls Take any pt B in PQ, with centre B, and radius equal to the given length, draw a ○ cutting RS in C, D Through A draw par^ls to BC, BD

Then the part intercepted on each of these is equal to the given length, since the opp sides of a par^m are equal

There are two, one, or no solutions according as the circle of construction cuts, touches, or does not meet RS, *i e* according as the given length is greater than, equal to, or less than the perp distance between the two parallels

6 Bisect ∠BAC by AO meeting BC in O Through O draw OE par^l to AC to meet AB in E, and OD par^l to AB to meet AC in D Then AEOD is a par^m by construction

Also ∠EOA=∠OAD [*Th* 11]=∠OAE

EO=EA ADOE is a rhombus [*Th* 21]

7 Let ABC be an equilat △, bisect the angles at B and C by BO, CO, through O draw OD, OE par^l to AB and AC respectively meeting BC in D and E Then, by Theors 11, 16, △ODE is equiangular to △ABC, so that ODE is equilat Again ∠DOB=alt ∠ABO=∠OBD,

OD=BD Similarly OE=EC

BD=DE=EC

8 (i) Through the given pts P, Q, R draw BPC, CQA, ARB par^l to QR, RP, PQ respectively Then from the par^ms RQPB, RQCP, we have BP=RQ=PC [*Th* 21]

Thus P is the mid-pt of BC Similarly for Q, R

(ii) Construct a △ABE such that AB, BE are equal to the lengths of the given sides and AE is double the length of given median

Complete the par^m ABEC Let BC, AE meet in D

Then AC=BE, and AD=½AE [*Th* 21 *Cor* 3]

ABC is the required △

(iii) See Fig to III p 97

Let BC be the given side Take two thirds of each of the given medians, hence construct a △BOC

Complete par^m BOCK and produce KO to A so that OA=OK

Then, by Ex 1, p 64, OY and OZ bisect AC and AB, and the rest of the proof follows as in III

(iv) See Fig to III p 97

Take two-thirds of each of the medians and construct the △KOC Complete the par^m KCOB and proceed as in (iii)

PART II

Page 101.

1 (i) Draw a line AB to represent 1 yard On AB describe a square ABCD Divide AB, AD each into 3 equal parts, and through the pts of division of each line draw parallels to the other line Then the square is now divided into compartments each of which represents 1 square foot There are 3 rows each containing 3 squares, the square representing 1 *square yard* contains 3×3 or 3^2 squares representing *square feet*

(ii) and (iii) are done similarly

4. Consider a square in the plan each of whose sides is one inch The area represented is (5×5) sq in

Hence 1 sq in represents 25 sq miles

. 6 sq in represent 150 sq miles

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The majority of Examples 1–10 are too simple for solution A few are given as specimens

3 By the extension of Theor 23 given on p 101,

the area $=(0 8\times3 5)$ sq in $=2 8$ sq in

On drawing the figure on squared paper ruled to tenths of an inch it will be found that there are 8 rows each containing 35 small squares Thus the number of squares is 280 But the area of each small square is one-hundredth of a sq inch Hence the area is 280 hundredths of a sq inch *i e* 2 8 sq in

6 The area $=(1 6\times2 1)$ sq in $=3 36$ sq in

By drawing a figure, it will be seen that the number of squares each equal to $\frac{1}{100}$ of a square inch is 16×21, *i e* 336 Thus the area $=336$ hundredths of a sq inch or 3 36 sq in

8 Here $a=7$ ft, $b=6$ ft, area $=(7\times6)$ sq ft

9 $a=2500$ metres, $b=4$ metres; area $=(2500\times4)$ sq m

10 $a = \frac{1}{4}(1760 \times 3)$ ft., $b = \frac{1}{12}$ ft.,

$$\text{area} = \frac{1760 \times 3}{4 \times 12} \text{ sq. ft.} = 110 \text{ sq. ft.}$$

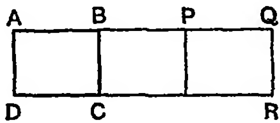
11 Since *the area of a rectangle = (length) × (breadth)*, we have *breadth = (area) ÷ (length)*.

Thus the breadth $= \frac{30}{6} = 5$ cm.

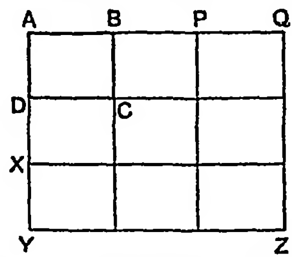
12 As in Ex. 11, *length = (area) ÷ (breadth)*.

Thus the length $= \frac{3.9}{1.5} = 2.6''$.

13 (i)



In the Figure, ABCD represents the original rectangle, AQRD the rect. obtained by trebling the length AB. It will be seen that the new rectangle is 3 times the area of the original rectangle.



(ii) ABCD represents the original rectangle, AQZY the new rectangle obtained by trebling AB and AD. It will be seen that the new rectangle is 9 times the area of ABCD.

The general rule is:

Each time we multiply *either* the length *or* the breadth of a rectangle by any factor, the area must also be multiplied by that factor.

14 The length and breadth of the garden are 36 yds. and 25 yds. respectively. ∴ its area $= (36 \times 25)$ sq. yds. $= 900$ sq. yds.

The new area is 1200 sq. yds.,

∴ new length $= \frac{1200}{25} = 48$ yds.

Since 10 yds. are represented by 1 inch, ∴ the length on the plan $= 4.8''$.

15 The actual length and breadth are

(6.5×20) metres and (4.5×20) metres, *i.e.* 130 m. and 90 m.

Hence area $= (130 \times 90)$ sq. m.

16 The area of the *plan* $= 3\ 2 \times 4\ 5 = 14\ 4$ sq cm

14 4 sq cm represent 1440 sq yds

1 sq cm represents 100 sq yds

1 cm 10 yds

Thus the scale is 10 yds to the centimetre

17 Actual length $= 3\ 25 \times 100 = 325$ ft

actual breadth $= \frac{52000}{325} = 160$ ft

breadth on plan $= \frac{160}{100} = 1\ 6''$

18 It will be noticed that the small rectangle on the right of the figure, whose dimensions are 10 ft by 5 ft, can be fitted exactly into the cavity in the upper portion of the figure Thus the area is equal to that of a rectangle whose length is 30 ft and breadth 20 ft, and is (30×20) sq ft *i e* 600 sq ft

19 From the right hand upper corner of the figure cut off the rectangle whose length is 24 ft and breadth 12 ft This can be fitted exactly into the hollow portion of the remaining figure to make up a rectangle whose dimensions are 24 ft by 48 ft Thus the area of original figure

$= (24 \times 48)$ sq ft $= 1152$ sq ft

20. It will be seen that the dimensions of the unshaded rectangle in the centre are obtained by subtracting twice the width of the border from those of the whole rectangle

Thus the inner rectangle has length $= 10$ ft, and breadth $= 5$ ft, and its area is 50 sq ft

But area of whole rectangle $= (10 \times 15)$ sq ft, or 150 sq ft

Thus area of shaded portion $= 150 - 50$, or 100 sq ft

21 To obtain the dimensions of the outer rectangle add twice the width of the border to those of the inner rectangle We thus get its length $= 15$ ft, and breadth $= 12\ 5$ ft

area of outer rectangle $= (15 \times 12\ 5)$ sq ft $= 187\ 5$ sq ft

Also area of inner rectangle $= (7 \times 4\ 5)$ sq ft $= 31\ 5$ sq ft

area of border $= (187\ 5 - 31\ 5)$ sq ft $= 156$ sq ft

22 Each of the rectangles at the four corners has its length $= \frac{1}{2}(15 - 5)$ ft $= 5$ ft, and breadth $= \frac{1}{2}(12 - 5)$ ft $= 3\ 5$ ft

sum of their areas $= 4(5 \times 3\ 5)$ sq ft $= 70$ sq ft

But area of whole rectangle $= (15 \times 12)$ sq ft $= 180$ sq ft

area of shaded portion $= (180 - 70)$ sq ft $= 110$ sq ft

23 Consider the dimensions of the four shaded figures at the corners of the original rectangle.

Then length $=\frac{1}{2}(30-18)$ ft $=6$ ft, breadth $=\frac{1}{2}(20-8)$ ft $=6$ ft.

$\therefore$ sum of their areas $=4(6\times 6)$ sq. ft $=144$ sq. ft.

also area of central rectangle $=(18\times 8)$ sq. ft $=144$ sq. ft.

$\therefore$ total shaded area $=(144+144)$ sq. ft $=288$ sq. ft.

24 Consider the four small squares into which the original square is divided by the dotted lines. Each is divided into two equal portions one shaded and one plain. Thus by addition the whole area of the shaded portion

$=\frac{1}{2}$(area of whole square) $=72$ sq. ft.

25 As in Ex. 24, whole shaded area $=\frac{1}{2}$ whole rectangle

$=\frac{1}{2}(10\times 15)$ sq. ft $=75$ sq. ft.

Page 105.

1 (i) Area $=(5.5\times 4)$ sq. cm. $=22.0$ sq. cm.

(ii) Area $=(2.4\times 1.5)$ sq. in. $=3.6$ sq. in.

2 Apply Prob. 12. Draw DK, BM perp. to AB, AD respectively.

By measurement DK $=1.36''$, $\therefore$ Area $=2.5\times 1.36=3.400$ sq. in.

This is only approximate since the length obtained by measurement is only correct to the nearest hundredth of an inch, and in finding the area any error in the measurement of one line is multiplied by the number of units in the other line. Thus in the above DK may be any length greater than $1.355''$ and less than $1.365''$, and in taking its length as $1.36''$ we must expect an error not exceeding $0.005''$, and there is a consequent error in the area not greater than (2.5×0.005) sq. in. or 0.0125 sq. in.

By measurement, BM $=2.27''$

$\therefore$ area $=$ AD . BM $=(1.5\times 2.27)$ sq. in. $=3.405$ sq. in.

Thus average of the two results $=\frac{1}{2}(3.400+3.405)$ sq. in.

$=3.4025$ sq. in.

3 By Prob. 12, draw a par^{m} having sides 6 cm., 5 cm. and included angle $=50°$.

By measurement, the perp. distance between the two long sides is 3.82 cm., representing 19.1 metres.

Thus area $=(19.1\times 30)$ sq. m. $=573$ sq. m. (approx.)

Again, by measurement, perp distance between the two short sides is 1 6 cm, representing 23 metres.

Thus area $=23 \times 25 = 575$ sq m (approx)

Hence the average result is 574·0 sq m

4 Altitude of par$^m = \frac{4\ 2}{2\ 8} = 1\ 5''$

Draw AB $= 2\ 8''$ and a line parallel to it at a distance of $1\ 5''$ With centre A, radius $=2''$, draw an arc cutting this line in D Then complete the par^m as in Prob 12

5 Altitude $= \frac{3\ 86}{2} = 1\ 93''$

Draw two parallel lines $1\ 93''$ apart Take a pt A in one line as centre, and, with radius $=2''$, cut the lines in B, D respectively Cut off DC on the second line $=2''$

Then ABCD is the rhombus

By measurement each of the acute angles $=75°$

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1. (i) Area $=\frac{1}{2}(24 \times 15)$ sq ft $=180$ sq ft

(ii) Area $=\frac{1}{2}(4\ 8 \times 3\ 5)$ sq in $=8\ 4$ sq in

(iii) Area $=\frac{1}{2}(160 \times 125)$ sq m $=10000$ sq m $=1$ hectare

2 (i) By measurement, perp on BC is 3 6 cm (to the nearest mm)
area $=\frac{1}{2}(8\ 4 \times 3\ 2)$ sq cm $=13\ 44$ sq cm (approx)

(ii) By measurement, perp on AB $=4\ 5$ cm (to nearest mm)
area $=\frac{1}{2}(6\ 8 \times 4\ 5)$ sq cm $=15\ 3$ sq cm (approx)

(iii) By measurement, perp on BC $=6\ 3$ cm (to nearest mm)
area $=\frac{1}{2}(6\ 5 \times 6\ 3)$ sq cm $=20\ 48$ sq cm (approx)

3 If we consider BC as the base the altitude of the $\triangle$ is seen to be AC, for ACB is a rt angle

area $=\frac{1}{2}$ BC $\times$ CA

When $a=6$ cm, $b=5$ cm, area $=\frac{1}{2}(6 \times 5)$ sq cm $=15$ sq cm

Draw CK perp to AB By measurement

AB $=7\ 8$ cm, CK $=3\ 8$ cm (each to the nearest mm)

area $=\frac{1}{2}(7\ 8 \times 3\ 8)$ sq cm $=14\ 82$ sq cm (approx)

The error is 0 18 sq cm in 15 0 sq cm

$\therefore$ error per cent $= \frac{0\ 18}{15} \times 100 = 1\ 2$

4 Area of rt.-angled $\triangle = \frac{1}{2}(2.8 \times 4.5)$ sq. in. $= 6.3$ sq. in.
By measurement, $c = 5.30'$, $p = 2.38''$
$$\text{area} = \tfrac{1}{2}(5.30 \times 2.38) \text{ sq. in.} = 6.307 \text{ sq. in.}$$
$$\text{error per cent} = \frac{0.007}{6.3} \times 100 = 0.1 \text{ (nearly)}$$

5 (i) Since $area = \frac{1}{2}\,(base) \times (altitude)$
$$altitude = \frac{2(area)}{base}$$
$$= \frac{2 \times 80}{20} = 8 \text{ in.}$$

(ii) As above, $base = \dfrac{2(area)}{altitude}$
$$= \frac{2 \times 10.4}{1.6} = 13 \text{ cm.}$$

6 By measurement, perp. $= 2.24''$
$$\text{area} = \tfrac{1}{2}(3.0 \times 2.24) \text{ sq. in.} = 3.36 \text{ sq. in.}$$

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1 (i) The $\triangle^s$ are on the same base BC and between the same par[ls] BC, XY. Now use Theor. 26.

(ii) The $\triangle^s$ are on the same base XY and between the same par[ls] XY and BC. Now use Theor. 26.

(iii) Add the $\triangle$AXY to each of the equal $\triangle^s$ BXY, CXY.

(iv) Take the $\triangle$XKY from each of the equal $\triangle^s$ BXY, CXY.

2 The two $\triangle^s$ are on equal bases and of the same altitude; hence, by Theor. 26, they are of equal area.
Divide the base into 3 equal parts by Prob. 7 and join the pts. of section to the vertex. The three $\triangle^s$ so formed have equal bases and a common altitude.

3 In the Fig. of Theor. 21, Cor. 3, the $\triangle$AOD $=$ $\triangle$DOC, being on equal bases AO, OC, and having a common altitude.
Similarly $\triangle$DOC $=$ $\triangle$COB $=$ $\triangle$BOA.

4 The $\triangle$ABX $=$ $\triangle$ACX, and $\triangle$YBX $=$ $\triangle$YCX. [*Th.* 26.]
Hence, by subtraction, $\triangle$ABY $=$ $\triangle$ACY.

5. See Ex 9 on p 60
The $\triangle^s$ ADX, ABX have a common base AX and equal altitudes DQ, BP, $\therefore$ they are equal in area by Theor 25
Similarly for $\triangle^s$ CDX, CBX

6 Let $\triangle$ ABC have sides AB, AC bisected in X, Y respectively,
then $\triangle$ BXY = $\triangle$ AXY, and $\triangle$ CXY = $\triangle$ AXY, [*Th* 26]
$\therefore$ $\triangle$ BXY = $\triangle$ CXY,
$\therefore$ XY is par[l] to BC [*Th* 27]

7 Let P, Q be the mid-pts of the slant sides AD, BC of trapezium ABCD Join PC, QD and AC, BD
Then $\triangle$ DPC = $\frac{1}{2}$ $\triangle$ DAC, $\triangle$ DQC = $\frac{1}{2}$ $\triangle$ DBC [*Th* 26]
Also $\triangle$ DAC = $\triangle$ DBC [*Hyp* and *Th* 26], $\therefore$ $\triangle$ DPC = $\triangle$ DQC,
$\therefore$ PQ is par[l] to DC [*Th* 27]

8 Par[m] AY = half par[m] AC,
and $\triangle$ AZB = half par[m] AY [*Th* 25 *Cor*]
$\therefore$ $\triangle$ AZB = one fourth par[m] AC

9 $\triangle^s$ AXB, BYC are each half of the given par[m] [*Th* 25 *Cor*]

10 Through P draw XY par[l] to AB or DC
Then $\triangle$ APB is half par[m] AY,
and $\triangle$ DPC is half par[m] XC,
that is, sum of $\triangle^s$ APB, DPC is half par[m] AC

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1 Draw a $\triangle$ whose sides are 3 7″, 2 0″, 1 9″ Taking 2 0″ as base, the altitude by measurement is 1 14″
Hence area of plan = $\frac{1}{2}$(2 0 × 1 14) sq in
But 1 sq inch represents 100^2 sq yds
$\therefore$ area of enclosure = $\frac{1}{2}$(2 0 × 1 14) × 100^2 sq yds
= 11400 sq yds

2 Draw two lines 6 2 cm and 7 2 cm respectively inclined at angle of 45° Completing the $\triangle$ and taking 7 2 cm as base, the altitude by measurement is 4 4 cm (to nearest mm)
Hence area of plan = $\frac{1}{2}$(7 2 × 4 4) sq cm (approx)
But 1 sq cm represents 20^2 or 400 sq m
$\therefore$ area of enclosure = $\frac{1}{2}$(7 2 × 4 4) × 400 sq m
= 6336 sq m (approx)

3 Since *base* × *altitude* = 2(*area of triangle*), [*Th* 25]

$$\text{altitude} = \frac{2 \times 6\,6}{5\,5} = 2\,4 \text{ cm}$$

Thus the locus of A is one of the two st lines par[l] to BC and at a distance of 2 4 cm on each side of it

With centre B, radius = 2 6 cm, draw an arc cutting one of these lines in A Join CA By measurement CA = 5 1 cm

4 $$\text{Altitude} = \frac{2 \times 3\,06}{3} = 2\,04'$$

Hence the locus of A is one of the two st lines par[l] to BC and at a distance 2 04′ from it

At C draw CA making ∠CBA = 68° and cutting one of these lines in A Join AC

By measurement, AC = 2 20′

5 For a graphical representation of the result see p 321

6 Since the equal sides contain supplementary angles the two △[s] can be placed having one side in common and the other two equal sides in same st line Thus we have two △[s] of same altitude on equal bases

Let the △[s] be ACB, ACD with the side AC common and the sides BC, CD in the same st line

If the △[s] are congruent, ∠ACD is equal to one of the ∠[s] of the △ABC, and it can be seen by Theor 8 that this is only possible when ∠ACB = ∠ACD, that is, when the two supplementary angles are rt ∠[s]

7 Bisect the base BC of the △ABC at D, and draw DK at rt ∠[s] to BC and meeting the par[l] to BC through A in K Then, by Theor 4, the △KBC is isosceles, and by Theor 26 it is equal in area to △ABC

8 Let P, Q, R, S be the mid-pts of AB, BC, CD, DA Let PQ RS cut BD in X, Y

Then	par[m] PXYS = 2 △SPB	[*Th* 25 *Cor*]
	= △ASB	[*Th* 26]
	= ½ △ADB	[*Th* 26]
Similarly	par[m] QXYR = ½ △BCD	

, by addition, par[m] PQRS = ½ quad[l] ABCD

9. △BQC = ½ △ABC [*Th* 26] = △ARC

Take away from each the △XQC

2 With Fig of Theor 28 (ii), $\triangle ABC = \frac{1}{2}(11 \times 17)$ sq ft
$\triangle ADC = \frac{1}{2}(9 \times 17)$ sq ft
quad^l ABCD $= \frac{1}{2}(20 \times 17)$ sq ft $= 170$ sq ft

3 Area of plan $= \frac{1}{2}(3.4 + 2.6) \times 8.2 = 24.6$ sq cm
But 1 sq cm represents 25 sq m
area of enclosure $= (24.6 \times 25)$ sq m $= 615$ sq m

4 Apply Prob 8 to draw first the $\triangle DAB$ and then $\triangle DCB$
By measurement, offsets from A and C are 2.4″, 1.6″ respectively
area of quad^l $= \frac{1}{2}(1.6 + 2.4) \times 4.2 = 8.4$ sq in

5 First draw the $\triangle DAB$, and then the $\triangle DCB$
By measurement, DB $= 8.5$ cm, offset from C to BD $= 4.1$ cm (to nearest mm)
area of $\triangle BCD = \frac{1}{2}(8.5 \times 4.1)$ sq cm $= 17.43$ sq cm (approx)
Area of *rt-angled* $\triangle DAB = \frac{1}{2}(3.6 \times 7.7)$ sq cm $= 13.86$ sq cm
area of quad^l $= 31.29$ sq cm (approx)

6 By measurement, DC $= 2''$, height $= 1.73''$ Now use Theor 28

7 Construct a $\triangle BPC$ such that BC $=$ CP $= 5$ cm, BP $= 6$ cm
Produce BP to A so that PA $= 3$ cm, complete par^m CPAD
Then ABCD is the req^d trapezium
By measurement, height $= 4$ cm Now apply Theor 28

8 Let diag BD cut diag AC at rt $\angle$s at X Then the offsets of B, D are BX, DX respectively, then sum $=$ BD
area $= \frac{1}{2}$ AC $\times$ (sum of offsets of B, D) $= \frac{1}{2}$ AC $\times$ BD
$= \frac{1}{2}$(product of diagonals)

9 Let the diags AC, BD of given length cut one another at the given angle at any point in each Through A and C draw PAS, QCR par^l to BD and through B and D draw PBQ, SDR par^l to AC Then the fig PQRS is a par^m whose sides and angles are the same at whatever point AC and BD intersect
Now $\triangle ABC = \frac{1}{2}$par^m APQC, and $\triangle ADC = \frac{1}{2}$par^m ASRC
quad^l ABCD $= \frac{1}{2}$ of the constant par^m PQRS

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1 (i) Area of $\triangle$ AED $=\frac{1}{2}(3\times 5)$ sq cm $=7.5$ sq cm
Area of $\triangle$ ADC $=\frac{1}{2}(4\times 6)$ sq cm $=12$ sq cm
Area of $\triangle$ ABC $=\frac{1}{2}(2\times 6)$ sq cm $=6$ sq cm
whole area $=25.5$ sq cm

(ii) $\triangle^s$ AED, BCD have equal bases and equal altitudes, and are therefore equal in area Hence

area of fig AEDCB $=\triangle$ ADB $=\frac{1}{2}(6\times 5.2)$ sq cm $=15.6$ sq cm

2 (i) Area of square AEDB $=2\frac{1}{2}\times 2\frac{1}{2}=6.25$ sq in
Area of $\triangle$ DCB $=\frac{1}{2}(2.5\times 2.16)$ sq in $=2.7$ sq in
whole area $=8.95$ sq in

(ii) $\triangle$ AXD $=\frac{1}{2}(1\frac{1}{4}\times 2\frac{1}{2})$ sq in $=1.5625$ sq in
$\triangle$ CYB $=\frac{1}{2}(1\frac{3}{4}\times 2)$ sq in $=1.75$ sq in
Trap^m DXYC $=\frac{1}{2}(2\frac{3}{4}\times 4\frac{1}{2})$ sq in $=6.1875$ sq in
whole area $=9.5$ sq in

			sq m	sq m
3.	$\triangle$ AXF $=\frac{1}{2}$	AX $\times$ XF	$=\frac{1}{2}\times 50\times 60$	$=1500$
	$\triangle$ AYB $=\frac{1}{2}$	AY $\times$ YB	$=\frac{1}{2}\times 120\times 50$	$=3000$
	$\triangle$ CYB $=\frac{1}{2}$	CY $\times$ YB	$=\frac{1}{2}\times 60\times 50$	$=1500$
	$\triangle$ CDZ $=\frac{1}{2}$	CZ $\times$ ZD	$=\frac{1}{2}\times 30\times 80$	$=1200$
	trap^m DZYE $=\frac{1}{2}$	YZ $\times$ (DZ + EY)	$=\frac{1}{2}\times 30\times 120$	$=1800$
	trap^m EYXF $=\frac{1}{2}$	XY $\times$ (EY + FX)	$=\frac{1}{2}\times 70\times 100$	$=3500$

$\therefore$, by addition, whole area $=12500$ sq m

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1. (i) As in Ex 7, p 64 the figure is a par^m Now, by Theor 4, the $\triangle^s$ PAS, PBQ, RCQ, RDS are congruent Hence PQ = PS = RQ = RS, and the figure is a rhombus

(ii) By Ex 8, p 110, the area of
PQRS $=\frac{1}{2}$(area of ABCD) $=\frac{1}{2}$AB $\times$ BC
Now, by Theor 20, QS = AB
Similarly PR = BC
$\therefore$ area of PQRS $=\frac{1}{2}$ PR $\times$ QS
$=\frac{1}{2}$(product of the diags of the rhombus)

(iii) See Ex 8, p 113

2 Let ABCD be a par^{m}, O the middle pt of the diag BD Draw any line through O meeting AB, CD in E and F respectively Then $\triangle^{s}$ EOB, FOD are identically equal [*Th* 14, 17]

fig AEFD $= \triangle$ ADB $=$ half the par^{m}

(i) Through O and the given pt P draw a line to cut a pair of par^{l} sides

(ii) and (iii) may be solved on the same principle

3 (i) Let PXQ drawn par^{l} to AD meet DC in Q and AB in P Then $\triangle^{s}$ BXP, CXQ are identically equal [*Th* 14, 17] the area of the trapezium is equal to that of par^{m} APQD

(ii) Trapezium ABCD $=$ par^{m} APQD $= 2 \triangle$ AXD [*Th* 25, *Cor*]

4 As in Ex. 8, p 113, the area $= \frac{1}{2}(3\cdot0 \times 2\cdot2)$ sq in $= 3\cdot3$ sq in

See Ex 9, p 113

5 Draw AB $= 8\cdot0$ cm and a st line par^{l} to it at a distance $3\cdot0$ cm from it With centres A, B, radii $3\cdot2$ cm, draw arcs cutting this line in D, C

Area of par^{m} $=$ AB $\times$ (perp dist between AB, DC) $= (8 \times 3)$ sq cm
Area also $=$ AD $\times$ (perp dist between AD, BC) $= (3\cdot2 \times x)$ sq cm
where $x =$ perp dist between AD, BC in cm

$3\cdot2x = 8 \times 3$, whence $x = 7\cdot5$

6 Construct a $\triangle$ AOB, having AB $= 2\cdot5''$, AO $= 1\cdot7''$, BO $= 1\cdot2''$

Produce AO, BO to C and D, so that OC $=$ OA, OD $=$ OB

Then by Theors 4, 13 the figure ABCD is a par^{m}

By measurement, perp dist between AB, DC $= 1\cdot44''$

area of par^{m} $= 2\cdot5 \times 1\cdot44 = 3\cdot6$ sq in

7 Let ABCD be the par^{m}, and O the intersection of its diags

It is easy to shew that $\triangle$ AOB $= \frac{1}{4}$ par^{m} ABCD

Hence the $\triangle$ AOB is of constant area, and, its base being fixed, its altitude must also be constant

Thus O moves so that its perp dist from AB is constant

its locus is one of two fixed st lines par^{l} to AB and on either side of it

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1. By Theor. 29, $c^2 = a^2 + b^2$, hence

   (i) $c^2 = 3^2 + 4^2 = 9 + 16 = 25$, $\quad c = 5$ cm

   (ii) $c^2 = (2.5)^2 + 6^2 = 6.25 + 36 = 42.25$, $\quad c = \sqrt{42.25} = 6.5$ cm

   (iii) $c^2 = (1.2)^2 + (3.5)^2 = 1.44 + 12.25 = 13.69$, $\quad c = \sqrt{13.69} = 3.7''$

2. (i) $b^2 = c^2 - a^2 = (3.4)^2 - (3.0)^2 = 2.56$, $\quad b = \sqrt{2.56} = 1.6''$

   (ii) $a^2 = c^2 - b^2 = (5.3)^2 - (4.5)^2 = 7.84$, $\quad a = \sqrt{7.84} = 2.8$ cm

3. Let AB represent the ladder, A its foot, and C the foot of the wall Then $AB^2 = AC^2 + CB^2$

   $$AB = \sqrt{AC^2 + BC^2} = \sqrt{81 + 1600} \text{ ft} = 41 \text{ ft}$$

   For the plan 2 mm to 1 foot may be taken

4 The required distance $= \sqrt{33^2 + 56^2}$ mi $= \sqrt{4225}$ mi $= 65$ mi
   Take 1 inch to 10 miles for the plan

5 Let S, A, B mark the positions of the station and the two ships Then $AB^2 = AS^2 + SB^2$, from rt.-angled $\triangle$ ASB

   $$AB = \sqrt{(6.0)^2 + (1.1)^2} \text{ km} = \sqrt{37.21} \text{ km} = 6.1 \text{ km}$$

   Take 1 cm to 1 km for the plan

6. Let AB represent the ladder, A its foot, and C the foot of the wall Then $AC^2 = AB^2 - BC^2$

   $$AC = \sqrt{65^2 - 63^2} \text{ ft} = \sqrt{256} \text{ ft} = 16 \text{ ft}$$

   To test this by measurement, the $\triangle$ would be drawn by Prob 10, p 83 Take 1 cm to 10 ft for the plan

7 From rt.-angled $\triangle$ ABC, $AB^2 = AC^2 - BC^2$

   $$AB = \sqrt{73^2 - 55^2} \text{ m} = \sqrt{18 \times 128} \text{ m} = 48 \text{ m}$$

   For the plan take 1 cm to 10 m and use Prob 10, p 83

8 Let O mark his starting point, and A, B, C his three stopping places Then if CD is drawn par[l] to BA, meeting OA at D, from the rt.-angled $\triangle$ ODC, $OC^2 = OD^2 + DC^2$

   $$OC = \sqrt{7^2 + 24^2} \text{ mi} = 25 \text{ mi}$$

   For the plan 1 inch to 10 miles may be taken

9. Name the successive turning pts B, C, D, E Produce BA, DE to meet in K Then

   $$AK = CD - AB = 55 \text{ m}, \text{ and } EK = BC - DE = 48 \text{ m}$$

   Then $AE^2 = AK^2 + KE^2$, from rt.-angled $\triangle$ AKE

   $$AE = \sqrt{55^2 + 48^2} \text{ m} = \sqrt{5329} \text{ m} = 73 \text{ m}$$

10 Let AB, CE represent the two walls, D the foot of the ladder, AD, DC its first and second positions

Then from rt-angled $\triangle$ ABD $BD=\sqrt{50^2-48^2}$ ft $=14$ ft,

and from rt-angled $\triangle$ CDE, $DE=\sqrt{50^2-14^2}$ ft $=48$ ft

$$BD=BD+DE=62 \text{ ft}$$

For the plan proceed as follows

(Scale 1 cm to 10 ft) Let D a point in the horizontal line XY represent the foot of the ladder Draw two lines par[d] to XY, and above it at distances of 4 8 cm and 1 4 cm respectively With centre D, and radius 5 0 cm, cut the first par[d] at A and the second at C Draw AB CE perp to XY

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1 Let BD be the diagonal of the given sq ABCD
Then $BD^2=BC^2+CD^2=2BC^2=2$ (given square)

2 $c^2=AD^2-DB^2$ and $b^2=AD^2+DC^2$,
, by subtraction, $c^2-b^2=BD^2-DC^2$

3 By Ex. 2,

$$AZ^2-ZB^2=AO^2-OB^2,$$
$$BX^2-XC^2=BO^2-OC^2$$
$$CY^2-YA^2=CO^2-OA^2,$$

by addition, $(AZ^2+BX^2+CY^2)-(ZB^2+XC^2-YA^2)=0$,

that is, $AZ^2+BX^2+CY^2 = ZB^2+XC^2+YA^2$

4 $BQ^2=BA^2+AQ^2$, and $CP^2=CA^2+AP^2$
, adding, $BQ^2+CP^2=(AB^2+AC^2)+(AQ^2-AP^2)=BC^2+PQ^2$

5 In the $\triangle$ BAC, rt angled at A let the medians be BQ, CP.
Then $BQ^2=BA^2+AQ^2$, and $PC^2=PA^2+AC^2$

$$4(BQ^2+PC^2)=4(BA^2+AC^2)-4PA^2+4AQ^2$$
$$=4BC^2 \quad + AB^2+ \quad AC^2$$
$$=4BC^2 \quad + BC^2 \qquad =5BC^2$$

6 Let AB CD be sides of the two squares, at B draw BE perp to AB and equal to CD Join AE
Then $AE^2=AB^2+BE^2=AB^2+CD^2$

7 See Prob 10, p 83 Let AB P be sides of the two given squares
Then $BC^2=AB^2-AC^2=$ difference of the given squares.

8 Let AB be the given st line

From B draw BD perp to AB and from A draw AC, making BAC one-fourth of a right angle From C, the intersection of AC and BD, draw CX, making $\angle$ACX equal to $\angle$BAC

Since $\angle$XCA $=\angle$XAC, XA $=$ XC

And because $\angle$BXC $=$ sum of $\angle$s BAC, ACX,

$\angle$BXC is half a rt $\angle$

And angle at B is a rt $\angle$ $\angle$BCX is half a rt $\angle$ [*Th* 16]

$\angle$BXC $=\angle$BCX, BX $=$ BC

Hence the square on XC is double of the square on XB [*Th* 29]

that is, the square on AX is double of the square on XB

9 Let AB be the given st line At B draw BC, making $\angle$ABC $=45°$ With centre A, radius equal to the side of the given square, describe a circle cutting BC in C Draw CD perp to AB

Then in $\triangle$CDB, $\angle$BDC $=90°$, $\angle$DBC $=45°$, $\angle$BCD $=45°$,

DB $=$ DC AD^2 + DB^2 = AD^2 + DC^2 = AC^2 = given square

10 On substituting the given values of the sides, it will be found that the formula $c^2=a^2+b^2$ is satisfied by the data of (i) and (ii) but not by those of (iii)

11 AB^2 = AC^2 + CB^2 = 2AC^2, since AC = CB

Let the diags of sq on AB intersect at O, and let CE be a diag of sq on AC Then since AOBC, ABCE are par^{ms}, it will be seen that $\triangle$AOB $=\triangle$AEC Now AB$^2=4\triangle$AOB, AC$^2=2\triangle$EAC, whence AB^2 = 2AC^2

AB^2 = 2AC^2 = (2×4) sq in, AB $=\sqrt{8}=2\,83''$

12. Draw AB $=6$ cm Bisect AB at O and with centre O, radius OA, describe a $\bigcirc$ Through O draw COD perp to AB and meeting the circle in C, D Then ACBD is the req^d square

By Theor 4 the sides can be proved equal, and as in Prob 10 the angles can be proved right angles the figure is a square

Let AC $=$ CB $=x$ cm

$$AB^2=AC^2+CB^2=2x^2, \qquad 2x^2=36 \text{ sq cm}$$

$$x=\sqrt{18}\text{ cm}=4\,24\text{ cm}$$

Also area $=x^2$ sq cm $=18$ sq cm

13 Let AC be a diag of square ABCD Let AB $=a$ units

$$AC^2=AB^2+BC^2=2a^2 \qquad AC=a\sqrt{2}$$

Req^d diag $=50\sqrt{2}=50\times 1\,4142=70\,71$ m

14 Let BD be perp from B on AC Then, by Theor 17,

$$AD=DC=m \text{ units}$$

Now $AB^2=AD^2+BD^2$, $\quad 4m^2=m^2+p^2$, whence $p=m\sqrt{3}$

15 $a^2+b^2=(m^2-n^2)^2+(2mn)^2=m^4+2m^2n^2+n^4=(m^2+n^2)^2=c^2$

When $n=1$, $m=2$, we have $a=3$, $b=4$, $c=5$,
$n=1$, $m=3$ $\quad a=8$, $b=6$, $c=10$,
$n=1$, $m=4$ $\quad a=15$, $b=8$, $c=17$,
$n=5$, $m=6$ $\quad a=11$, $b=60$, $c=61$,
and so on

16. (i) $AB=\sqrt{12^2+9^2}$ cm $=15$ cm Also $DC=(25-9)$ cm $=16$ cm

$$AC=\sqrt{12^2+16^2} \text{ cm} =20 \text{ cm}$$

(ii) $AD=\sqrt{50^2-30^2}$ in $=40''$ $\qquad DC=\sqrt{41^2-40^2}$ in $=9''$

$$BC=BD+DC=39''$$

(iii) $BD^2=BA^2-AD^2=c^2-p^2$, $\qquad BD=\sqrt{c^2-p^2}$

Similarly $DC=\sqrt{b^2-p^2}$,

$$a=BC=\sqrt{b^2-p^2}+\sqrt{c^2-p^2}$$

17 $c^2-BD^2=AB^2-BD^2=AD^2=AC^2-CD^2=b^2-CD^2$

Let $BD=x$ cm $\qquad CD=(51-x)$ cm

$37^2-x^2=20^2-(51-x)^2$, whence $x=35$

$p^2=37^2-35^2=144$, $\qquad p=12$

area $=\frac{1}{2}ap=306$ sq cm

18 We shall find exactly as in Ex 17

(i) $BD=7\frac{16}{17}''$, $\qquad p=4\frac{4}{17}''$.

(ii) $BD=9\,6$ ft, $\qquad p=7\,2$ ft

(iii) $BD=13\frac{28}{41}$ cm, $p=6\frac{6}{41}$ cm

(iv) $BD=5$ yds, $\qquad p=12$ yds

Whence the areas can be obtained

19 In the 2^{nd} position let $OQ = x$ cm

$\therefore x^2+(4{\cdot}0)^2=PQ^2=(5{\cdot}6)^2-(3{\cdot}3)^2=42{\cdot}25.$

Whence $x^2=26{\cdot}25$ and $x=5{\cdot}12$ nearly.

20. Let CK be the perp. from C on AB.

Then the area of the $\triangle$ is $\frac{1}{2}$CK.AB, and also $\frac{1}{2}$ AC.CB.

$\therefore$ CK.AB=AC.CB; i.e. $pc=ab$

Hence $\dfrac{1}{p}=\dfrac{c}{ab}$, $\therefore \dfrac{1}{p^2}=\dfrac{c^2}{a^2b^2}=\dfrac{a^2+b^2}{a^2b^2}=\dfrac{1}{b^2}+\dfrac{1}{a^2}$

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1. Let ABCD be the given square. Join AC. Then, by Theor. 16, $\angle$CAB$=45°$.

Through B draw BE par' to AC to meet DC produced in E.

Then ABCD, ABED are equal in area [*Th.* 24].

We have $AC^2=AB^2+BC^2=50$ sq. cm.; $\therefore$ $AC=5\sqrt{2}=7{\cdot}1$ cm.

2. Use Prob 12, p. 87, taking any convenient angle for the included angle.

With centre A, radius AB draw an arc cutting DC in E. Join AE. Through B draw BF par' to AE, cutting DC produced in F. Then ABFE is a rhombus and is equal to ABCD by Theor. 24.

3. By Theor. 21, each of the par^{ms} AEKH, KGCF, ABCD is bisected by AC.

$\therefore$ $\triangle$AEK+$\triangle$KGC=$\triangle$AHK+$\triangle$KFC.

Taking these equal areas from the equal $\triangle^s$ ABC, ADC respectively, we have fig. EKGB=fig. HDFK.

Place HK, KG in the same st. line. Through H draw HA par' to KE to meet BE produced in A. Draw AK meeting BG produced in C. Complete par^{m} ABCD, and let EK produced cut CD at F. Then HKFD is the par^{m} required.

4. Produce DE to G, making EG equal to AB, or 6 cm. Complete the par^{m} FEGH. Join HE, and produce it to meet CD produced at K. Complete the par^{m} HCKL. Produce FE to meet KL at M. Then EMLG is the required rectangle; and EM=4 cm.

5. Produce BC to K, so that CK=2·7". Through K draw KL par' to BA and meeting AD produced in L. Draw LC to cut AB produced in P. Through P draw PQ par' to

AD and meeting LK produced in Q Produce DC to cut PQ in R Then KCRQ is equal to ABCD by Ex 3

By measurement it will be found that CR is 1 6″, and that this is true whatever value we give to A

$$\text{Also } 1\,6 \times 2\,7 = 1\,8 \times 2\,4$$

Hence we deduce that in equiangular par^{ms} of equal area the product of a pair of adjacent sides is constant

6 First construct a rect equal in area to $\triangle ABC$ (Use Prob 17, taking $D = 90°$) Now proceed as in Ex 4 to construct a second rect equal to the first and having one side equal to 5 cm

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1. Use Probs 11 and 18 If ADX be the resulting $\triangle$, the perp from X on AD $= 10\,6$ cm (to nearest mm)

   Hence area of quad$^l = \frac{1}{2}(10\,6 \times 4\,5) = 23\,85$ sq cm (approx)

2 Use Prob 8 to construct first $\triangle ADB$ and then $\triangle DBC$ Through C draw CX par^l to DB meeting AB produced in X Join DX Then ADX is the req^d $\triangle$

   By measurement, perp from X on AD $= 4\,67''$ (approx)

   area of quad$^l = \frac{1}{2}(4\,67 \times 3\,6)$ sq in $= 8\,406$ sq in

3. At A, B draw AE, BC each equal to AB, making the $\angle^s$ BAE, ABC each $= 108°$

   With centres E, C and radii equal to AB draw arcs cutting in D Then ABCDE is equilateral

   Use the Fig and construction of Prob 18, Cor By measurement we have XY $= 8\,9$ cm, perp from D on XY $= 6\,2$ cm

   area $= \frac{1}{2}(6\,2 \times 8\,9)$ sq cm $= 27\,59$ sq cm (approx)

4 Draw the Fig on the req^d scale by two applications of Prob 8

   If CX drawn par^l to DB meets AB produced in X, we shall find by measurement that perp from X on AD $= 13\,3$ cm (to the nearest mm)

   area of plan $= \frac{1}{2}(7\,8 \times 13\,3)$ sq cm $= 51\,87$ sq cm (approx)

   But 1 sq cm represents $(50)^2$ sq m

   area of field $= 51\,87 \times 2500 = 129675$ sq m (approx)

5 Join AD Through C draw CE par^l to AD to meet BA (or BA produced) in E Then, as in Prob 18,

$$\triangle EBD = \triangle ABC$$

6 Let CAB be the given △ on base AB Through A draw AD perp to AB and equal to the given altitude, and through C draw CE par[l] to AB meeting AD in E Join DB, and draw EF par[l] to DB meeting AB in F Join DF, EB

Then △ DAF = △ EAB = △ CAB [*Prob* 18]

7 Join BX Through A draw AD par[l] to BC meeting BX in D Then △ CDB = △ ABC Join XC Draw DF par[l] to XC to meet BC in F, and join XF

△ XBF = △ DBC [*Prob* 18] = △ ABC

8 Through C and D draw CE, DF par[l] to BX and AX respectively meeting AB produced in E and F, then EXF is the req[d] △

9. Divide the base into n equal parts [*Prob* 7], and join the pts of division to the vertex

12 As the method is quite general, it will be sufficient to take a particular case Let ABC be a triangle from which it is required to cut off a fifth part by a st line through a pt X in AB Take BD a fifth part of BC [*Prob* 7] Join AD, and through X draw XE to meet BC in E, so that

△ XBE = △ ABD [*Ex* 5]

14. With the construction of Prob 18, make the △ DAX equal to the given quadrilateral Take AY equal to one-fifth of AX; join DY Then △ DAY = one fifth of △ DAX, that is, of the quadrilateral The method is quite general

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3. Let PM, QN be the ordinates of P (4, 3) and Q (12, 7)

Let R be the mid-pt of PQ, and RL its ordinate

Then RL = $\frac{1}{2}$(PM + QN) [*Ex* 9, p 65]

Again OL − OM = ML = LN [*Th* 22] = ON − OL,

whence OL = $\frac{1}{2}$(OM + ON)

5. Draw ordinates from (0, 6), (0, 12) to meet the line joining O to P (18, 15)

Then OP is trisected by these ordinates, and the ordinates are respectively 5 and 10 units [*Th* 22 *Cor*]

Thus the req[d] coords are (6, 5), (12, 10)

7. If P (x, y) represents any one of the points, we have clearly

$$OP^2 = x^2 + y^2 \qquad [Th\ 29]$$

8 See Ex. 1, p 133

9 If P, Q, R represent the pts in the order given, it will be found, as on p 133, (Example 1) that $PQ^2=PR^2$ Thus the pts form an isosceles $\triangle$ whose vertex is P

10 If P represents anyone of the pts, it will be found in each case (by the method of Ex 7) that $OP^2=25$

11 On plotting the points it will be found that in each case the distance is the hypotenuse of a rt-angled $\triangle$ whose sides are a and b

12 The lines are the diagonals of a square

13 If P, Q, R are the points in the order given, it will be found that $PQ=QR=13$

Let QR meet the x-axis in S, then, by congruent $\triangle^s$, PR and OS bisect each other [*Th* 17]

14 The fourth vertex is obviously at O, and since the diagonals bisect each other, the coords of their intersection are those of the mid-pt of the line joining (0, 0) and (14, 10) See Ex 3

15 As in the Ex on p 133, it may be shown that the length of each side is 13 The intersection of the diagonals is the mid-pt of the line joining (0, 0) and (18, 12) See Ex 3

16 The locus is the perpendicular bisector of the distance between the two points [*Prob* 14], and cuts the axes at the pts (4, 0), (0, -4)

18 Each side is clearly equal to $\sqrt{2}$ inches, (Ex 13, p 124) and the figure is a square whose area is 2 sq in The second figure is a square on a side of 1″

19 It will be found that each triangle has a base of 14 units (lying on the x-axis) and height 10 units Hence the area is $\frac{1}{2}(10\times 14)$, or 70, units of area.

20 To facilitate measurement it will be found convenient to take half an inch as the unit Take the origin as vertex

In (i) base=6 units, height=3 units

$$\text{area}=\tfrac{1}{2}(3\times 6)=9 \text{ units of area}$$

21 To find the area, choose that side of the $\triangle$ which is par[1] to an axis as base, and measure the altitude along the other axis

22 These $\triangle^s$ are *rt angled* Take the two sides which are par[1] to the axes as base and altitude respectively

23. If A, B, C, D are the points, the lengths of AB, BC can be found as in Ex 1, p 133 To find the area, draw parallels to the x-axis through B and D, and parallels to the y-axis through A and C The area of the surrounding rectangle thus formed is 9×15, or 135, units of area The areas of the corner $\triangle^s$ thus formed are easily found to be 30, 30, 6, 6 units of area respectively Hence the area of ABCD is $135 - 72$, or 63, units of area

24 See Theorem 28

25 Follow the method of the example worked out on p 134

26 See solution of Ex 23 In this case the area is easily seen to be twice that of a triangle whose base is 23 and altitude 5

27. Let P, Q, R, S be the points in the order given, then by Theor 29,

$$PQ=\sqrt{5^2+12^2}=13, \quad RQ=\sqrt{8^2+6^2}=10,$$
$$SR=\sqrt{12^2+9^2}=15$$

By measurement (or calculation) $SP=8\ 24$

Let SR meet OY in T, and let RM be the ordinate of R

Then area of fig $OTRQ = \text{trap}^m\ OTRM + \triangle RMQ$
$$=\tfrac{1}{2}\times 4(3+6)+\tfrac{1}{2}\times 6\times 8 \qquad [\textit{Th } 28]$$
$$=18+24=42 \text{ units of area}$$

And area of $\triangle OPQ=\tfrac{1}{2}OP\ OQ=\tfrac{1}{2}\times 5\times 12$
$$=30 \text{ units of area}$$

28 Here $BC=9$ obviously, and as in Ex 1, p 133 we easily find that $AB=10$, $CD=17$ $AD=12\ 7$ by measurement

Let CB and DA produced meet in E The coords of A shew that DA bisects the angle between the axes, hence it easily follows that E is the point $(-10, -10)$

$$\text{Area of } ABCD = \triangle ECD - \triangle ABE$$
$$=\tfrac{1}{2}\times 23\times 15-\tfrac{1}{2}\times 6\times 14$$
$$=130\ 5 \text{ units of area}$$

29 If P, Q, R, S, O are the given points, the sides can all be found as in Ex 1, p 133

If SQ meets OY in T the req[d] area is the sum of the areas of the $\triangle$[s] PTQ, QSR, STP

30 Through A draw a par[l] to OX meeting the ordinates from B and C in D, E respectively

Then $\triangle ABC =$ trap[m] ECBD $- \triangle AEC - \triangle ADB$

$$= \tfrac{1}{2}(8 \times 8) - \tfrac{1}{2}(1 \times 4) - \tfrac{1}{2}(4 \times 7)$$

$$= 16 \text{ units of area}$$

And each unit represents 100×100 sq yds ,

area of field $= 160{,}000$ sq yds

Again, $BC = \sqrt{8^2 + 6^2} = 10$ units

corresponding side of field $= 10 \times 100$ yds

Let p be the perp from A on BC, then

$$\tfrac{1}{2}\, p \times BC = 16 \times 100 \times 100,$$

whence $p = 320$ yds

31 Each side of the figure is the hypotenuse of a right-angled $\triangle$ whose sides are 6, 14 Hence each side $= \sqrt{6^2 + 14^2} = 15\ 23$

(i) The circumscribing square contains 20^2 units of area, from this we must subtract four rt-angled $\triangle$[s] each of which contains $\tfrac{1}{2}(6 \times 14)$, or 42, units of area Thus the given square contains $400 - 4 \times 42$, or 232, units of area

(ii) By the method of page 120, we find that the given square consists of four equal rt-angled $\triangle$[s], each of which contains 42 units of area, together with a central square on a side of 8 units

the square contains $168 + 64$, or 232 units of area

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1 Of AB, AC let AB be the greater Produce AX to E making $XE = AX$ Join EB Then by Theor. 4, the $\triangle$[s] AXC, BXE are congruent, and $EB = AC$, and $\angle BEX = \angle CAX$

Also from $\triangle ABE$, since $AB > BE$,

$$\angle BEX > \angle BAX, \text{ i.e. } \angle CAX > \angle BAX$$

From this it easily follows that $\angle CAX > \angle CAP$, and ∴ AP lies within angle CAX

Again $\angle^s$ CAD, BAD are complements of $\angle^s$ ACD, ABD respectively, and $\angle ACD > \angle ABD$ ∴ $\angle CAD$ is less than $\angle BAD$, from which we have $\angle CAD$ is less than $\angle CAP$, and ∴ AP lies without angle CAD

Thus AP is intermediate in position between AD, AX

∴, by Ex 8, p 55, it is also intermediate in magnitude

2 Let ABC be the given $\triangle$, AB being greater than AC, and AH the bisector of the vertical $\angle BAC$ Draw CLK perp to AH, cutting AH in L, and AB in K

Then by Theor 17 the $\triangle^s$ ALK, ALC are congruent, and

$$\angle AKL = \angle ACL$$

(i) Now $\angle AKC$ = sum of $\angle^s$ KBC, KCB [*Th* 16]

To each add $\angle ACK$

then twice $\angle ACK$ = sum of $\angle^s$ ABC, ACB,

∴ $\angle ACK$ = half sum of $\angle^s$ ABC, ACB

(ii) As before, $\angle ACK$ = sum of $\angle^s$ KBC, KCB

To each of these add $\angle KCB$

then $\angle ACB = \angle KBC$ + twice $\angle KCB$

∴ twice $\angle KCB$ = difference of $\angle^s$ ACB, KBC,

that is, $\angle KCB$ = half difference of $\angle^s$ ACB, ABC

3. This follows at once from Ex 2 (ii) and Ex 8, p 43

4. Let the given diff be equal to AB and the hypotenuse equal to K From A draw AE making with BA produced an $\angle$ equal to half a rt $\angle$ From centre B, with radius equal to K, describe a circle cutting AE or AE produced in the points C, C′ From C and C′ draw perps CD, C′D′ to AB, and join CB, C′B Then either of the $\triangle^s$ CDB, C′D′B will satisfy the given conditions

NOTE If the given hypotenuse K be greater than the perpendicular drawn from B to AE, there will be *two* solutions If the line K be equal to this perpendicular, there will be *one* solution, but if less, the problem is *impossible*

5 (i) Let AB be the given base, X the difference of the $\angle$s at the base, and K the difference of the remaining sides

Draw BE, making the $\angle$ABE equal to half the $\angle$X See Ex 2

From centre A, with radius equal to K, describe a circle cutting BE in D and D' Let D be the point of intersection *nearer to* B

Join AD and produce it to C

Draw BC, making $\angle$DBC equal to $\angle$BDC

Then CAB is the $\triangle$ required

NOTE This problem is possible only when the given difference K is greater than the perpendicular drawn from A to BE

(ii) Let AB be the given base, K the sum of the remaining sides, and X the difference of the $\angle$s at the base Make the $\angle$ABD equal to half the $\angle$X, draw BE perp to BD, and from centre A and with radius equal to K describe a circle cutting BE in E Join AE, and at B make $\angle$EBC equal to $\angle$AEB Then ACB is the required $\triangle$, [*Ex.* 2,] since, if AE meets BD at F, it may be shewn that CB=CF

6. Let AB be the given base, and K the sum of one side and the perp from the vertex to the base

Bisect AB at X draw XH perp to AB and equal to K

Join AH at A draw AC making $\angle$HAC=$\angle$AHX and cutting XH in C

Then ACB is the $\triangle$ required, for CH=CA

7 See Ex 8, p 123

8 (i) If O is without the $\angle$DAB and its opp vert $\angle$, then OA is without the par^{m} ABCD therefore the perp drawn from C to OA is equal to the sum of the perp^{s} drawn from B and D to OA [See Ex 13, p 65]

Now the $\triangle$s OAC, OAD, OAB are upon the same base OA, and the altitude of the $\triangle$OAC with respect to this base = the sum of the altitudes of the $\triangle$s OAD, OAB

the $\triangle$OAC = the sum of the $\triangle$s OAD, OAB

(ii) Let BX, DY, CZ be the perp^{s} on AO Then drawing CH perp to BX we may shew, by Theor 17, that $\triangle$sADY, CBH are congruent, and BH=DY Hence

CZ=BX−DY.

Now proceed as in (i)

9 In the quad[l] ABCD, draw BO par[l] to the diag AC, and CO par[l] to AB, then DBO is the required $\triangle$, and ABOC is a par[m] Also the perp from D on BO is equal to the sum of the perp[s] from D on AC and from B on AC

$$\therefore \triangle DBO = \triangle DAC + \triangle ABC,$$

since these $\triangle$[s] have equal bases

10 Let BC be the given base and ABC the $\triangle$ in any of its positions Then since the area of the $\triangle$ is constant and the base is known the length of the altitude is also constant Let O be the intersection of the medians, then as in Ex 4, p 109 it is easy to prove

$$\triangle BOC = \triangle COA = \triangle AOB,$$

$$\therefore \triangle BOC = \text{one-third of } \triangle BAC = \text{constant}$$

Now these $\triangle$[s] have same base hence the altitude of $\triangle$ BOC is one third of the altitude of $\triangle$ BAC and is constant Hence the locus of O is a st line par[l] to BC and at a fixed distance from it

11 Let ABC be the given $\triangle$ on base BC, and DE the given st line Through A draw AF par[l] to BC meeting DE in F, and join FB, FC Then FBC is the required $\triangle$ [*Th* 26] If ED is par[l] to BC, the solution is only possible when DE passes through A In this case any pt in DE may be taken as the vertex of the required $\triangle$, and the number of solutions is unlimited

12 Let E, F be the mid-pts of AB, CD Then, by Theor. 20, EF = BC and is constant, and E is a fixed point Hence the locus of F is a semicircle, with centre E and radius BC

PART III

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1 By Theor 29, $OB^2=OD^2+DB^2$ $OB=\sqrt{3^2+4^2}=5$ cm

2 With the Fig of Theor 31, OD the perp from O on AB bisects it and $AB=2DB$

But $DB=\sqrt{OB^2-OD^2}=\sqrt{13^2-5^2}=12''$ $AB=24''$

3. Let AB represent the chord 1 6″ long, and OD the perp from the centre, then D is the mid-pt of AB, and $BD=0\ 8''$

Now $OD^2=OB^2-BD^2$ $OD=\sqrt{1-0\ 64}=\sqrt{0\ 36}=0\ 6''$

Similarly for the chord 1 2″ long

4. Here $OB=4$ cm, $BD=3$ cm

$OD=\sqrt{OB^2-BD^2}=\sqrt{7}$ cm $=2\ 6$ cm (to the nearest mm).

5 Here $BD=35''$, $OB=37''$ $OD=\sqrt{37^2-35^2}=12''=1$ ft

6 Here $BD=1\ 2''$, $OB=1\ 3''$, and $OD=\sqrt{OB^2-BD^2}=0\ 5''$

area of $\triangle AOB=\frac{1}{2}\ AB\times OD=\frac{1}{2}(2\ 4\times 0\ 5)$ sq in $=0\ 6$ sq in

7 With centres P and Q, radii 1 7″, describe ⊙s cutting in O Then O is the centre of the req^d ⊙

Draw OX perp to PQ then, by Theor 31, X is middle pt of PQ.

Thus $OX^2=OP^2-PX^2$ $OX=\sqrt{1\ 7^2-1\ 5^2}=\sqrt{0\ 64}=0\ 8''$

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1 Let O be the common centre, ABCD the st line cutting the inner ⊙ in B, C, and the outer ⊙ in A, D Draw OX perp to ABCD Then $BX=CX$, and $AX=DX$ [*Th* 31]

difference AB = difference CD

2 By Theor 31, the ∠s AMC, BMC are both rt ∠s

by Theor 2, AMB is a st line

AB is the line of centres, and CD the common chord of the two ⊙s, and we have proved that AB is perp to CD and passes through its middle pt

3 Let the bisector of $\angle$ BAC cut BC in D Then in $\triangle^s$ ADB, ADC, it follows that DB = DC and $\angle$ ADB = $\angle$ ADC [*Th.* 4]
That is, AD bisects BC at rt $\angle^s$, and passes through the centre

4 The required locus is the st line bisecting at rt $\angle^s$ the st line joining the two given pts [See Prob 14, p 91]

5 Let A, B be the two given points, and PQ the given st line Join AB, and bisect it at rt angles by a st line which meets PQ at O Then O is the centre of the required $\odot$ [See Ex 4] Impossible when PQ is at rt angles to AB or AB produced

6 Let A, B be the given points, and R the given radius With centres A and B, and radii equal to R describe $\odot^s$ cutting at O Then O is the centre of the required $\odot$
Impossible when the given radius is less than half AB

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1 Bisect AB, BC at rt $\angle^s$ by DO, EO respectively Then O is the centre of the req^d $\odot$ [*Th* 32] By construction, OEBD is a rect, and OE = BD = 0 8″, and EB = 1 5″
Also $OB = \sqrt{OE^2 + EB^2} = \sqrt{(0\,8)^2 + (1\,5)^2} = \sqrt{2\,89} = 1\,7''$

2 Draw AB = 6 cm From D the mid-pt of AB draw DO perp to AB and equal to 3 cm Then a $\odot$ with centre O radius OA is the $\odot$ req^d Also $OA^2 = OD^2 + DA^2$
Whence $OA = 3\sqrt{2}$ cm $= 4\,2$ cm (to the nearest mm)

3 Let AB be the diameter, O the centre, CD a chord of given length Bisect CD at E, then $\angle OED = 90°$

$$OE^2 = OD^2 - ED^2 = 16 - 4 = 12,$$

whence $OE = \sqrt{12} = 2\sqrt{3}$, or 3 46 cm (nearly)

4. With the notation of the Fig of III p 143 we have

$$OP = 26'', \quad PR = 24'', \quad PO' = 25''$$
$$OR^2 = OP^2 - PR^2, \text{ whence } OR = 10''$$
Also $RO'^2 = PO'^2 - PR^2$, whence $RO' = 7''$
Thus $OO' = OR + RO' = 17''$

5 Let AB, CD be the longer and shorter chords respectively, E, F their mid-pts and O the centre Then OE being perp to AB [*Th* 31], and also to the par^l line CD [*Th* 14], must pass through F [*Th* 31] Hence EOF is a st line

Now $OE^2=OA^2-AE^2$, whence $OE=2 5''$ Similarly $OF=6 0''$

EF, which is the sum or difference of OE, OF, according as the chords are on opposite sides or the same side of O, is either 8 5″ or 3 5″

6. As in the last Ex, let $OE=x$ cm, and $OF=(x+1)$ cm

Now $$OE^2+EA^2=(\text{radius})^2=OF^2+FC^2$$
$$x^2+16 =(\text{radius})^2=(x+1)^2+9,$$
whence $x=3$ Thus $OA=\sqrt{3^2+4^2}=5$ cm

7. The reason is that the points (6, 5) and (6, −5) are *symmetrically opposite* with regard to the x-axis, which is the line of centres [See p 143, III]

8. The st line, through the centre, perp to one of the par[l] chords, is perp to the other [*Th* 14] And this st line bisects both chords [*Th* 31] Hence, the st line joining the middle pts of two par[l] chords passes through the centre

9. The st line, through the centre, perp to *one* of the par[l] chords, is perp to *all* of them [*Th* 14] And this st line bisects all the chords Hence it is the required locus

10 Let AB, CD be two chords bisecting one another at E If possible let some other pt O be the centre, then, by Theor 31, ∠OEA and ∠OEC are both rt ∠s, which is impossible Thus no other pt but E can be the centre, and each chord is therefore a diameter

11. The diagonals of a par[m] bisect one another, ∴ by the previous Ex their pt of intersection is the centre

12 Let the par[m] ABCD be inscribed in a ○ Then the diagonals AC, BD intersect in O, the centre of the ○ [*Ex* 11]

Hence the diags are equal, being diameters of the ○,

∴ the par[m] ABCD is a rectangle [See Ex 5, p 59]

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1. Since equal chords are equidistant from the centre, the locus is a circle, whose centre is the centre of the given circle and whose radius is equal to the distance of any of the chords from the centre

2 Let chords AB, CD cut in E Let F be the centre Draw FG, FH perp to AB, CD respectively Then the Δ^s FEG, FEH are congruent [*Th* 17] GF = HF, the chords AB, CD are equidistant from the centre, and are therefore equal [*Th* 34]

3 Take Fig of preceding Ex Then EG = EH But BG, HD, the halves of AB, CD [*Th* 31], are equal the whole or remainder BE = whole or remainder DE

4 With centre A, on the O^{ce} of the given O, and with radius equal to the required *length*, cut the given circle in B From the centre of the given circle, draw a st line perp to the required *direction* and equal to the distance of the centre from AB The chord drawn through the extremity of this st line, par[l] to the given direction, will be the chord required

5 Let O be the centre and AX, BY, OZ the perp[s] on PQ. Then OZ = half the sum or diff of AX and BY, according as A and B are on the same or opposite sides of PQ [*Ex* 9, p 65] the sum or difference of AX and BY = twice OZ = a constant

6 For a theoretical proof see Ex 1
Let AB be any such chord, F its mid-pt, O the centre
Then $OF^2 = OA^2 - AF^2 = (4\ 1)^2 - (0\ 9)^2 = 16$, $OF = 4\ 0$ cm

7 As in Fig of III p 143, we have $OR^2 = OP^2 - PR^2 = (3\ 7)^2 - (1\ 2)^2$, whence $OR = 3\ 5''$
Hence take O, O′ at a distance of 4″, measure off OR = 3 5″, and draw RP, RQ perp to OR on either side of it and each equal to 1 2′ Then P, Q are the pts of intersection
$PO'^2 = PR^2 + RO'^2 = (1\ 2)^2 + (0\ 5)^2$, whence $PO' = 1\ 3'$

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1 Let O be the centre, F the given pt Join OF, and through F draw AFB perp to OF Then AB is the least chord
For draw through F any other chord PFQ and let OX be the perp to it from the centre Then OXF is a rt $\angle^d$ Δ and its hypotenuse OF > OX Thus AB is further from the centre than PQ, AB < PQ

2 Bisect BC at D Through D draw DE perp to BC and meeting AB in E Then E is the req[d] centre

Since $(3\ 7)^2=(3\ 5)^2+(1\ 2)^2$, the ∠ACB is a rt ∠ [*Th* 30] DE is par[l] to AC and consequently bisects AB [*Ex* 1, p 64] Hence radius=EB=1 85″

3 See Theor 32, or Prob 25, p 193

4. Let XZY, X′Z′Y′ be two chords bisected at Z and Z′ in AB, of which Z is nearer than Z′ to C the middle pt of AB Let O be the centre Then OC, OZ OZ′ are respectively perp to AB, XY, X′Y′ Also OZ′>OZ>OC [*Th* 12 *Cor* 3] X′Y′<XY<AB Hence AB is the greatest length of XY, and XY increases as Z approaches C When Z coincides with A or B, XY vanishes

5. Let P be (2 4″, 1 8″), Q be (1 8″, 2 4″), O the origin Then it is found from Theor. 29 that OP=OQ=3 0″

Through P, Q draw PN, QM perp to OX Let R be mid-pt of PQ, and RS its perp on OX

Through P draw PK perp to QM

(i) Then PK=0 6″, QK=0 6″. $PQ^2=PK^2+QK^2=0\ 72$, whence PQ=0 85″ (nearly)

(ii) By Ex 3, p 134 the coords of R are

$\frac{1}{2}(2\ 4''+1\ 8'')$ and $\frac{1}{2}(1\ 8''+2\ 4'')$, namely, (2 1, 2 1)

(iii) Since R is the mid-pt of PQ, OR is the perp distance of O from PQ.

Hence $OR^2=OS^2+SR^2=8\ 82$, whence OR=2 97″ (nearly)

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1. Let P be the fixed point, AB the given st line, and let *any* one of the circles having its centre in AB and passing through P cut the line through P perp to AB again in Q. Then, by Theor 31, Q is the *image* of P in AB [see p 141]; that is, Q is a fixed point

2 Let AB be the common chord of two ⊙s whose centres are E, F, and let the st line par[l] to AB cut one ⊙ at P, Q and the other at X, Y Join EF, cutting PQ at O Then EF is perp to AB [III p 143] EF is perp to PQ [*Th* 14] OP=OQ, and OX=OY [*Th* 31] Hence PX=QY

3 Let the two ⊙s intersect in A, B Let CAD, EBF be par[l] lines cutting the one circle in C, E and the other in D, F Then the st line through the centre of ABEC, perp to the chords AC, BE, bisects these chords in P and Q, say Similarly the st line through the centre of ABFD perp to the chords AD, BF bisects these chords in X and Y, say But PQ is par[l] to XY [*Th* 14] PX=QY [*Th* 20] But CA is double of PA, and AD is double of AX CD is double of PX Similarly EF is double of QY, CD=EF

4 Let the two ⊙s intersect at A, B, and let PAQ, XAY be two st lines equally inclined to AB and terminated by the ⊙ces Through B draw P'BQ' par[l] to PQ Then P'Q'=PQ [*Ex* 3] Now the ∠XAB=the ∠P'BA, whence it may be shewn that XA=P'B [*Th* 34, *Converse*], and similarly AY=BQ' XY=P'Q'=PQ.

5 With the notation of the Fig of III, p 143, we have

$$OR=\sqrt{OP^2-PR^2}=\sqrt{37^2-12^2}=\sqrt{49\times 25}=35''$$

and $$RO'=\sqrt{O'P^2-PR^2}=\sqrt{20^2-12^2}=\sqrt{32\times 8}=16''$$

$$OO'=OR+RO'=51''$$

6. With the notation of the previous Ex (O being the centre of the larger ⊙), let $OR=x$ inches, $O'R=(2\,1-x)$ inches

Now $$OP^2-OR^2=PR^2=O'P^2-O'R^2,$$

$$(1\,7)^2-x^2=PR^2=(1\,0)^2-(2\,1-x)^2$$

Hence $x=1\,5$, and $PR=\sqrt{(1\,7)^2-(1\,5)^2}=0\,8''$

Thus the common chord$=2PR=1\,6''$, and its distances from the centres are $1\,5''$ and $0\,6''$ respectively

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1 Let A, B be the centres of the two ⊙s, and D, E the pts where AB (produced both ways) cuts the ⊙ces Let any other line PQ cut the ⊙ces at P and Q. Join QA, and produce it to cut the "A" circle at R Then AE>AQ [*Th* 37] To these unequals add the radii DA, RA respectively Then DE>RQ. But QR>QP [*Th* 37] DE>PQ.

Similarly, if AB cuts the ⊙ces at F, G, it may be shewn that FG<PQ.

2 Let C be the centre of the ○, A any pt on the ○^{ce}, ACB the diameter, AD, AF any other chords, and let ∠ACD be greater than ∠ACF

Then (i) AC+CD > AD [*Th* 11] But CD=CB, ∴ AB > AD

(ii) AC, CD=AC, CF, each to each, but ∠ACD > ∠ACF, ∴ base AD > base AF. [*Th* 19]

3 Let the two ○^{s}, whose centres are E and F, cut at A, B Let PAQ be the st line through A par^{l} to EF and terminated by the ○^{ces}, and let XAY be any other st line terminated by the ○^{ces}

From E, F draw EG, FH perp to XY, and from E draw EK perp to FH Then XY is double of GH, that is, double of EK [*Th* 31] Similarly PQ is double of EF But in the rt-angled △EKF, EF is greater than EK ∴ PQ is greater than XY

4 As in III, p 143, the pt of intersection will be the image of (8, −11) in the x-axis, and its coords are therefore (8, 11)

5 Let the points in the given order be A, B, C Then the radii of the ○^{s} are AC, BC $AC^2=15^2+8^2$, whence AC=17 Similarly $BC^2=6^2+8^2$, whence BC=10 Again the 2nd pt of intersection is the image of C in the line of centres AB, that is, the pt (0, −8)

6. It will be found that each of the angles APB, AQB, ... is 40° Hence we infer that angles in the same segment of a circle are equal (See Theor 39)

Taking the angle at O to be 70°, each of the angles APB etc will be found to be 35°, which suggests that the angle in a segment is half the angle subtended at the centre by the chord of the segment (See Theor 38)

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1. By Theor 39, ∠BAC=∠BDC=74°

By Theor 38, ∠BOC=2∠BDC=148°

And the sum of the equal ∠^{s} OBC, OCB=180° − 148°=32°,

∴ ∠OBC=16°

2 From $\triangle$ XDC, $\angle BDC = 180^\circ - (40^\circ + 25^\circ) = 115^\circ$. [*Th* 16]
by Theor 39, $\angle BAC = \angle BDC = 115^\circ$
By Theor 38, $\angle BOC = 2\angle BDC = 230^\circ$

3 From $\triangle$ BDC, $\angle BDC = 180^\circ - (43^\circ + 82^\circ) = 55^\circ$ [*Th* 16.]
by Theor 39, $\angle BAC = \angle BDC = 55^\circ$
Also $\angle BOC = 2\angle BDC = 110^\circ$,
$\angle OBC = \angle OCB = \frac{1}{2}(180^\circ - 110^\circ) = 35^\circ$.
Hence $\angle OBD = \angle DBC - \angle OBC = 8^\circ$,
and $\angle OCD = \angle DCB - \angle OCB = 47^\circ$.

4 Denote $\angle BAC$ by A
Then reflex $\angle BOC = 2A$ $\angle BOC = 360^\circ - 2A$
Now, by Theor 16, since $\angle OBC = \angle OCB$, we have
$$2\angle OBC = 180^\circ - \angle BOC$$
$$2\angle OBC = 180^\circ - (360^\circ - 2A) = 2A - 180^\circ$$
$$\angle OBC = A - 90^\circ$$

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2. By Theor 39, $\angle ADB = \angle ACB$, and $\angle BDC = \angle BAC$
Adding, $\angle ADC = \angle ACB + \angle BAC$
Thus $\angle ADC + \angle ABC = \angle ACB + \angle BAC + \angle ABC = 180^\circ$. [*Th* 16]

3 The opp $\angle^s$ of a par^m are equal, and if a circle can be described about the par^m they are together equal to two rt. $\angle^s$ each $\angle$ is a rt $\angle$

4 The $\angle ABC = \angle ACB = 180^\circ - \angle XYC$ [*Th* 14] XBCY is concyclic. [Converse of *Th* 40]

5 Let ABCD be a cyclic quadrilateral, having one side DA produced to E
Then the $\angle^s$ DAB, DCB together = two rt $\angle^s$, [*Th* 40]
and the $\angle^s$ DAB, BAE together = two rt. $\angle^s$
Hence $\angle BAE = \angle DCB$.

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1 Let the △ ABC be rt angled at C, and let P be any point on the circle whose diam is AB
Then ∠APB=90° [*Th* 41]=∠ACB [*Hyp*]
C is on the arc APB [*Th* 39, *Converse*]

2. Each of the ∠s PBA, QBA in a semicircle is a rt ∠ [*Th* 41]
PB, QB are in a st line

3 The line joining the vertex of an isosceles △ to the middle pt of the base is perp to the base [Ex 1, p 26] Hence the ○ on a side as diameter passes through the middle pt of the base [*Ex* 1]

4. Both the circles described on the sides of a △ as diameters must pass through the foot of the perp from the vertex on the base or base produced

5 The locus is a quadrant of the ○ whose centre is the pt of intersection of the rulers, and radius half the length of the rod [*Th* 41]

6. Let A be the fixed point, C the centre of the ○, and APQ any chord through A, meeting the ○ce at P, Q. Let X be the middle point of PQ Then CX is perp to PQ.

That is, the ∠AXC is a rt angle, and since AC is a fixed base, the point X lies on the ○ce of a ○ on AC as diam

(i) If A is external, the locus is that part of the ○ on AC which is intercepted within the given ○.

(ii) If A is on the ○ce, the locus is a complete ○ described on the radius AC as diam, and having internal contact with the given ○

(iii) If A is internal, the locus is a complete ○ falling within the given ○

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1. In the △ APB, the ∠APB is constant [*Th* 39]
And, by Theor 16, ∠PAB+∠PBA=180°−∠APB=constant

2. The ∠s QRS, QPS in segment QRPS are equal and the ∠s RQP, RSP in segment RQSP are equal and the opp vertical ∠s RXQ, SXP are equal

3 The $\angle$s APB, AQB, in the segments APB, AQB are constant [*Th.* 39]
Hence from $\triangle$ PBQ, the third $\angle$PBQ is constant. [*Th.* 16]

4. The $\angle$PBX $=\angle$PAX [*Th.* 39] $=$ vert. opp. $\angle$YAQ
$=\angle$YBQ [*Th.* 39].

5 The $\angle$AOB $=180^\circ-(\frac{1}{2}\angle$PAB $+\frac{1}{2}\angle$PBA$)$ and is constant [*Ex.* 1].
Hence locus of O is the arc of a $\odot$ on chord AB. [Converse of *Th.* 39]

6 Let the chords AB, CD intersect within the $\odot$ at E. Join AD
Let the centre be O
Then $\angle$AEC $=\angle$BAD $+\angle$ADC [*Th.* 16]
$=\frac{1}{2}\angle$BOD $+\frac{1}{2}\angle$AOC [*Th.* 38]
$=$ half the sum of the $\angle$s subtended at the centre by the arcs BD, AC, or the $\angle$ at the centre subtended by an arc equal to half the sum of the arcs BD, AC [*Th.* 43]

7 Let the chords AB, CD intersect without the $\odot$ at E. Join AD.
Then $\angle$AEC $=\angle$ADC $-\angle$DAB [*Th.* 16]
That is, $\angle$AEC $=$ the difference of the $\angle$s at the $\odot$ce subtended by the arcs AC, BD, or the $\angle$ at the centre subtended by half the difference of the arcs AC, BD

8. Let AB, CD two chords of a $\odot$ intersect at rt. $\angle$s at E.
Then by Ex. 6 the $\angle$AEC is equal to the sum of the $\angle$s subtended at the $\odot$ce by AC, BD
That is the sum of the arcs AC, BD subtend a rt. angle at the $\odot$ce, or the sum of the arcs is equal to a semi-circumference

9 Let the bisector of $\angle$APB cut the conjugate arc in Q.
Then $\angle$APQ $=\angle$BPQ. $\therefore$ arc AQ $=$ arc BQ [*Th.* 42].
$\therefore$ Q is the pt. of bisection of the conjugate arc AQB

10 The $\angle$AXY $=$ the $\angle$ subtended at the $\odot$ce by the sum of the arcs AQ, PB. [*Ex.* 6]
Similarly the $\angle$AYX $=$ the $\angle$ subtended at the $\odot$ce by the sum of the arcs QC, AP
But by hyp. the arcs AQ, PB $=$ the arcs QC, AP respectively
$\therefore$ $\angle$AXY $=\angle$AYX, $\therefore$ AX $=$ AY

11. The $\angle$AXY $=\angle$ABY $=\frac{1}{2}$B. And $\angle$AXZ $=\angle$ACZ $=\frac{1}{2}$C [*Th.* 39]
$\therefore$ $\angle$ZXY $=\frac{1}{2}(B-C)=90^\circ-\frac{A}{2}$ [*Th.* 16]

12 Let PA, PB cut the other ○ in Q and R Join BQ.

(1) Let Q and R be in PA, PB produced Then by Theor 16, ∠QBR = sum of ∠s BQA, BPA = sum of ∠s subtended at the Oces by AB = constant

(2) Let Q and R be in PA, PB Then ∠QBR = difference of ∠s BQA, BPA = difference of ∠s subtended at the Oces of the two ○s by AB = constant

(3) Let R be in PB and Q in PA produced Then ∠QBR = supplement of ∠s BQA, BPA = supplement of ∠s subtended at the Oces by AB = constant

13 Let AB, CD be pard chords of a ○ Then ∠ADC = ∠DAB [*Th* 14] arc AC = arc BD [*Th* 42] chord AC = chord BD [*Th* 45] And ∠CAB = supplement of ∠ACD [*Th* 14] = ∠ABD [*Th* 40] chord BC = chord AD [*Th* 42, 45]

14. By Theor 3, PX and QY subtend equal ∠s at the Oces of equal ○s Hence arc PX = arc QY Hence chord PX = chord QY [*Th* 45]

15 Each of the given lines = the common chord [*Ex* 13]

16. Since the chord AB is common to the two equal ○s, the arc AB in one = the arc AB in the other [*Th* 14], ∠APB = ∠AQB [*Th* 43] BP = BQ.

17. The chords BX, XA, AY, YC are equal, for each subtends at the Oce an ∠ equal to half the base ∠

Hence, if the base ∠s are each double of the vertical ∠, the pentagon is equilateral

18. Join PR, QR, and BR Then PR, QR shall be in one st line

For ∠PRB = the suppt of ∠PCB [*Th* 40]
= the suppt of ∠BAD [*Ex* 5, p 163]
= the suppt of ∠BRQ.
P, R, Q are collinear

19. Let P and X be on BC, Q on CA Then QX = QC [*Ex* 1, p 165]
∠QXC = ∠QCX = ∠PRQ, since PCQR is a parm [*Ex* 2, p 64]
∠QXP = ∠PRQ, or its suppt P, R, Q, X are concyclic [*Th* 40 Con]

20. Let Y and Z be the feet of the perps from B and C on CA and AB, respectively Then, as in Ex 19, P, Q, R, Y and P, Q, R, Z are concyclic But only one circle can pass through P, Q, R [*Th* 32] Hence the six pts P, Q, R, X, Y, Z are concyclic

21 If all the given Δ^s stand on a fixed base BC, and have a given vertical angle, they also have the *same circumscribed circle* [*Th* 39 Converse]

Take BAC, any one of these Δ^s, and let the bisector of the $\angle$A meet the circum-circle at X

Then $\angle$BAX $=\angle$CAX [*Hyp*], arc BX $=$ arc CX [*Th* 42]

X, being the middle point of the arc BC, is same for all triangles of the series

22 Draw CF perp to AE Then AE bisects the $\angle$BAC [*Th* 43] Hence $\angle$FCB $=$ half the diff of the $\angle^s$ at B and C [*Ex* 2, p 138] Now DE, EA are respectively perp to BC, CF

$\angle$DEA $=\angle$BCF [*Ex* 8, p 43]

$=$ half the diff of the $\angle^s$ at B and C

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1 Let AB be a chord of the outer $\bigcirc$, touching the inner $\bigcirc$ at M

Then if O be the common centre, OMA is a rt $\angle$ and AM $=$ MB

Hence $AM^2 = AO^2 - OM^2 = 16$ AM $= 4$ cm

And AB $=$ 2AM $= 8$ cm

The chords are all equal, since they are equidistant from the centre of the $\bigcirc$ in which they lie

2 Let AB be any such chord, M its middle point Then OMA is a rt $\angle$ $OM^2 = OA^2 - AM^2 = 0\ 36$ Thus OM $= 0\ 6''$ Thus all the mid-pts lie on a concentric circle, whose radius $= 0\ 6''$ Also since $\angle$OMA is always a rt $\angle$, each chord will be a tangent to this circle

3 With the notation of Ex 1, $AM^2 = (5)^2 - (2\ 5)^2 = 18\ 75$ Whence AM $= 4\ 33$ cm Thus AB $=$ 2AM $= 8\ 7$ cm (to nearest mm)

4 Since OPT is a rt $\angle$, $TP^2 = TO^2 - OP^2 = 144$ TP $= 12''$

5 With the figure of Theor 47, $TO^2 = (0\ 7)^2 + (2\ 4)^2 = 6\ 25$ Thus TO $= 2\ 5''$

6 Let O be the centre of a $\bigcirc$ touching AB and AC in B and C Then in the rt-angled Δ^s ABO, ACO, OB $=$ OC, OA is common, and $\angle^s$ ABO, ACO are rt $\angle^s$, [*Th* 46]

$\angle$OAB $=\angle$OAC [*Th* 18]

7 Let AO cut BC in D Then, by Theor 47, AB $=$ AC, and, by Ex 6, $\angle$BAD $=\angle$CAD Δ^s BAD, CAD are congruent [*Th* 4], whence BD $=$ DC and $\angle$BDA $=\angle$CDA

8. $\angle OPQ = \angle OTQ$ in segment OPTQ. Similarly $\angle OQP = \angle OTP$. But $\angle OPQ = \angle OQP$ [*Th.* 5]; ∴ $\angle PTQ = 2\angle OPQ$.

9. Let CD, BE be the two par^l tangents at the extremities of the diameter CAB; and DFE a tangent at F.
Then the $\triangle^s$ ABE, AFE are identically equal [*Th.* 47, Cor.]
∴ AE bisects $\angle$BAF. Similarly AD bisects $\angle$CAF
∴ DAE is a rt. $\angle$ [*Ex.* 6, p. 13].

10. The tangent at an extremity of a diameter is perp. to the diameter; ∴ the chords par^l to the tangent are also perp. to the diameter, and are bisected by it [*Th.* 31].

11. The required locus is the perp. to the given st. line through the given pt. [*Th.* 46].

12. The required locus is the st. line which is par^l to the two given st. lines and equidistant from them.

13. The required locus is the pair of bisectors of the angles between the two given st. lines. [See *Ex.* 6 and *Prob.* 15.]

14. Let ABCD be the quad^l, and P, Q, R, S the points of contact of the sides AB, BC, CD, DA.
Then AS=AP [*Th.* 47, *Cor.*] and DS=DR; ∴ by addition AD=AP+DR. Similarly BC=BP+CR.
Hence AD+BC=AP+BP+DR+CR=AB+DC.

The converse is: *If the sum of one pair of opposite sides of a quad^l is equal to the sum of the other pair then a circle may be inscribed in the figure.*

Let ABCD be a quad^l in which AB+CD=BC+DA. By bisecting the two $\angle^s$ ABC, BCD, describe a ○ to touch three sides AB, BC, CD [*Prob.* 26]. Then shall AD also touch this ○. For if not, let AD′ touch the ○ and cut CD at D′. Now by hyp., AB+CD=BC+AD. Also, by above, AB+CD′=BC+AD′. ∴ taking the differences of these equals, DD′=the difference of AD and AD′; hence either AD′=AD+DD′, or AD=AD′+DD′; which is impossible [*Th.* 11].

15. With the figure and notation of first part of Ex. 14, O being the centre of the ○, we have $\angle AOP = \angle AOS$ [*Th.* 47, Cor.] and $\angle BOP = \angle BOQ$; ∴ by addition, $\angle AOB = \angle BOQ + \angle AOS$. Similarly $\angle COD = \angle COQ + \angle DOS$.

Hence $\angle AOB + \angle COD = \angle BOC + \angle AOD$.

But these four $\angle^s$ together make up 360°. ∴ each pair = 180°, i.e. $\angle^s$ AOB, COD are supplementary.

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1 The point on the line of centres which is 1 7″ from the centre of the larger ○ and 0 9″ from the centre of the smaller ○ clearly lies on each ○ Also, by Theor 11, there can be no other pt of intersection, the ○s touch at this point

In general, when the distance between two centres is equal to the *sum* of the radii, the ○s touch at a pt between the two centres and the contact is *external*

When the distance between the two centres is equal to the *difference* of the two radii, the contact is *internal*

3 By Theor 29, $c=10$ cm Let the ○ with centre A cut AB in P, and BA produced in Q. Then BP=3 cm and BQ=17 cm

Hence ○s with centre B, and radii 3 cm and 17 cm, will touch the given ○ with external and internal contact respectively

4 Let the ○, centre P, touch the large ○ at Q, and the smaller ○ at R, and let the pt of contact of the two given ○s be C Then, by Theor 48, APQ, PRB, ABC are all st lines

$$AP+PB=AP+PR+RB=AP+PQ+BC=AQ+BC$$
$$=\text{sum of radii of given ○s}=\text{a constant}$$

5 Let P, Q be the mid-pts of AC, CB respectively, R the centre of the fourth ○, and D its pt of contact with the semicircle on AB Then, by Theor 48, CRD is a st line Since PRQ is isosceles, and C the mid-pt of PQ, we have RCQ=rt angle [*Th* 7]

Let r inches be the reqd radius Then $PR=r+1$, $CR=2-r$

Now $PR^2=CR^2+CP^2$ [*Th* 29] $(r+1)^2=(2-r)^2+1^2$

Whence $r=\frac{2}{3}''$

6 Let C be the pt of contact Then A, C, B are in a st line [*Th* 48] Because AC=AP, ∠ACP=∠APC And because BC=BQ, ∠BCQ=∠BQC But ∠ACP=∠BCQ. ∠APC=∠BQC AP, BQ are parallel

7 With the figure of Ex 6, let PT, QK (drawn in opp directions) be the tangents at P, Q. Then, as above, ∠s APC, BQC are equal

But ∠s APT, BQK are rt ∠s, ∠CPT=∠CQK, PT, QK are pard [*Th.* 13]

8. (i) The required locus is the st line through the centre of the given ○ and the given pt [*Th* 48]

(ii) Let O be the centre of the given ○, OP its radius On PO, or OP produced, take PC equal to the given radius of the circles which are to touch the given ○ Then the ○ with centre C and radius CP will touch the given ○ at P And OC=the sum or diff of OP and CP Hence the required locus is a circle with centre O and radius equal to the sum or diff of the radius of the given circle and the given radius of the touching circle

9 Let A be the centre of the given ○, B the given pt Let AB produced cut the given ○ in C and D The ○ described with centre B and radius BC or BD will touch the given ○ Hence there are two solutions except when B is on the ○ce of the given ○

10 Let A be the centre of the given ○ of radius b, P the given pt Take PC equal to a, either on PA (produced if necessary) or on AP produced The circles described with centre C and radius CP will touch the given circle at P but, if $a=b$, one of the two circles so described will coincide with the given circle

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1. $\angle$BAD$=\angle$FBD [*Th* 49]=72°

$\angle$EBD=supplement of $\angle$FBD=108°

$\angle$BCD$=\angle$EBD [*Th* 49]=72°

2 Each of the $\angle$s made by the tangents with the line joining their pts of contact is equal to the angle in the alt segment these $\angle$s are equal the tangents are equal [*Th* 6]

3. Let T'AT be the common tangent to the two ○s at A, AX being between AP and AT

Then $\angle$TAX$=\angle$APX [*Th* 49]

And, in (i), $\angle$TAX$=\angle$AQY

in (ii), $\angle$TAX$=\angle$T'AY$=\angle$AQY

in (i) and in (ii) $\angle$AQY$=\angle$APX

PX is par[l] to QY

4 Tangent at A to the ○ through O makes with AO an ∠ equal to ∠OBA in alternate segment But because O is centre of the other ○, OA = OB ∠OBA = ∠OAB AO bisects ∠ between AB and the tangent at A

5 The tangent at P makes with PAC an ∠ equal to ∠PBA = ∠ABD (or its supplement) = ∠ACD (or its supplement) tangent at P is par[d] to CD

6 Let A be pt of contact, AB chord through A, C the middle pt of arc cut off by AB, CM, CN perp[s] on the tangent at A and the chord AB Then ∠CAM = ∠ABC [*Th* 49] and

∠ABC = ∠CAB [*Th* 43] ∠CAM = ∠CAB

the △[s] CAM, CAN are identically equal [*Th* 17]

CM = CN

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2 From Theor 31 we know that if AB be any chord drawn through a pt A on the ○ and D its middle point, then ∠ODB is a rt ∠

This is true however near B approaches to A If B moves up to coincidence with A, the chord AB becomes the tangent AT, and D coincides with A, hence the ∠ODB becomes the ∠OAT and the theorem becomes

"If A be any point on a ○ and AT the tangent at A, then ∠OAT is a rt angle"

3. Let A, B be the centres of the two ○[s] in Fig 1, p 173, and let M be the mid-pt of the common chord PQ Let Q move up to coincidence with P Then whatever be the position of PQ we know that A, B, M are in one straight line This must be true in the final position when Q coincides with P, and M with each of them Hence if two ○[s], centres A and B, touch one another at P, then A, P, B are collinear

4. Let ABCD be the cyclic quad[l] having BC produced to F

Then ∠DCF = ∠BAD [*Ex* 5, p 163]

Now let C move along the circumference until it coincides with B Then secant BCF becomes the tangent at B, and the ∠DCF becomes the ∠DBT, where BT is the tangent at B Hence if BT be the tangent at B, and BD a secant through B, then ∠DBT = ∠BAD in alternate segment

5. In the Fig of p 181, let C move along the $\odot^{ce}$ until it coincides with B Then the chord CB becomes the tangent at B, and AC becomes the diameter AB Hence the $\angle$ between the tangent at B and the diameter through B is a rt angle

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1 (i) Two direct, and no transverse

(ii) Two direct, and a third at the point of contact

(iii) One, — at the point of contact

Let A be the centre of the 1 5" $\odot$, and B of the 1 0" $\odot$,

(i) The $\odot$s will intersect, and there will be two direct, and no transverse common tangents, for B will be within the $\odot$ of construction for transverse tangents

(ii) The $\odot$s will touch externally, and there will be two direct common tangents, and one transverse at the point of contact of the two $\odot$s For B will be on the $\odot$ of construction for transverse tangents

(iii) The $\odot$s will touch internally, and there will be one direct common tangent at the point of contact, since B will be on the $\odot$ of construction, and no transverse common tangents

(iv) The $\odot$s will neither touch nor cut each other, and there will be two direct and two transverse common tangents

2 No transverse common tangents exist With the Fig and notation of Prob 23, AB = 2·0", AC = 1 2, and ACB is a rt $\angle$, $DE^2 = BC^2 = AB^2 - AC^2 = 2\ 56$, $DE = 1\ 6''$

Similarly the other tangent = 1 6

3 With the Fig and notation of Prob 23, AB = 1 8", AC = 0 6", ACB is a right angle, $DE^2 = BC^2 = AB^2 - AC^2 = 2\ 88$, DE = 1 7", nearly

Similarly the other direct tangent = 1 7', nearly There is a transverse common tangent of unlimited length at the pt. of contact of the two $\odot$s.

4. Proceeding as before, the lengths of the common tangents are found to be 1 98" nearly To find the common chord proceed as in Ex. 6, p 155

5 Proceeding as in Prob 23, we shall obtain two direct and two transverse common tangents

6 In this case the ○ of construction is reduced to a point. Proceed thus —join AB the centres of the given ○ˢ, and draw AD, BE perp to AB, cutting the ○ᶜᵉˢ in D and E Join DE, which will be one direct common tangent [Proof by Theors 13, 20, and 46]

7. Let a pair of common tangents touch the greater ○ at D, D′, the smaller at E, E′, and cut one another at P

Then by Theor 47, Cor, $PD = PD'$, and $PE = PE'$

$$\text{for direct tangents} \qquad PD - PE = PD' - PE',$$

$$\text{and for transverse tangents} \quad PD + PE = PD' + PE',$$

$$\text{in either case} \qquad DE = D'E'$$

Or again, with the Fig of p 185, $DE = BC$, similarly $D'E' = BC'$, but $BC = BC'$, $DE = D'E'$ If the ○ˢ are equal, then the direct common tangents are equal [*Th* 20]

8. Let the direct common tangents DE, D′E′ touch the ○ˢ whose centres are A, B at D, E and D′, E′, and cut one another at P Join PB, BE, BE′ Then in the △ˢ PEB, PE′B, we have $BE = BE'$ and BP common, also the ∠ˢ PEB, PE′B are rt ∠ˢ [*Th* 46],

$$\angle EPB = \angle E'PB \; [Th\ 18]$$

That is, the centre B lies on the bisector of the ∠ between the common tangents Similarly the centre A lies on the same bisector the points A, B, P are collinear Similarly, if the transverse tangents cut at Q, then A, B, Q are collinear

9 Let B, C be the centres of the two given ○ˢ then BC passes through A [*Th* 48] Join BP, CQ

$$\text{Then } \angle BAP + \angle CAQ = \angle BPA + \angle CQA$$
$$= 90° - \angle APQ + 90° - \angle AQP$$
$$= 180° - (\angle APQ + \angle AQP) = \angle PAQ \; [Th\ 16]$$

Hence ∠ PAQ is half of two rt ∠ˢ, that is, the ∠ PAQ is a rt ∠

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1 See Theorem 32, or Problem 25, p 193

2 The centre lies on the line through A perp to PQ [*Th* 47]
On the perp to AB through its middle pt [*Th* 32]
Through A draw AO perp to PQ, draw CO bisecting AB at rt angles, and meeting AO at O With centre O and radius OA describe a ○, which will be the ○ required.

3. The centre lies on CA, or CA produced
Draw EO bisecting AB at rt angles, to meet CA at O The centre of the ○ must lie on both CA and EO, and will therefore be at O, and its radius will be OA or OB

4. Draw a line XY par^l to AB on the same side as P and at a distance 3 2 cm from AB, with centre P and radius 3 2 cm cut XY at Q and R With centres Q and R, and radii QP, RP respectively, describe ○s they will be the ○s required

5. Let A and B be the centres of the 3 0 cm and 2 0 cm ○s respectively With centres A and B, and radii (3 0+3 5) cm and (2 0+3 5) cm respectively, describe ○s cutting at P and Q then the two circles whose centres are P and Q, and whose radii are 3 5 cm, will touch each of the given ○s externally
Let r be the radius, then $3+r+2+r \not< 6$ [*Th* 11], $2r \not< 1$, or $r \not< \frac{1}{2}$ Hence the radius of the smallest ○ is 0 5 cm

6. Since the ⊙ touches both OA and OB, its centre is equidistant from these lines it must lie on one of the bisectors of the angles formed by OA, OB [*Prob* 15]
Draw a line par^l to OA, distant 1 2″ from it, to cut these bisectors at P and Q Then the ○s of radius 1 2″ whose centres are at P and Q will each touch OA and OB There will be four ○s, for the parallel line can be drawn on either side of OA

7. Draw a line XY par^l to AB, distant 2 5 cm from it, and on the same side of it as the centre of the given ○ Draw a ○ of radius (3 5+2 5) or 6 cm, concentric with the given ○, to cut XY at P and Q these pts will be the centres of the two ○s, of radii 2 5 cm, which can be drawn to touch the given ○ and the given line AB [Proof by (iv) and (v), p 188]

8 Let the transversal cut the two par^l lines PA, QB at A and B respectively, then the centre of the required ○ is equidistant from PA and AB, it must lie on the bisector of the ∠PAB Similarly it must lie on the bisector of ∠QBA it is at the point of intersection of these bisectors, and the radius will be the perp let fall from this point of intersection on any one of the lines
Since the centre can be the pt of intersection of either the internal or external bisectors there will be two ○s And these will be equal, for in each case the radius is half the distance between the par^l lines

9 Let C be the centre of the given ○, PQ the given line, and A the given pt in PQ.

At A draw AF perp to PQ, then the centre of the required ○ must lie in AF

Draw BD, a diam^r of the given ○, perp to PQ. Join A to one extremity D of the diam^r, cutting the O^{ce} at E Join CE, and produce it to cut AF at F

Then F is the centre, and FA the radius of the required ○, and a second solution is got by joining AB [Proof by *Th* 46, 48]

10 Let PQ be the given st line, and E the given point on the ○ of which C is the centre Draw the diameter BD perp to PQ. Join DE (or BE), and produce it to meet PQ at A Draw AF perp to PQ, and join CE, producing it to cut AF at F Then F is the centre of the required ○

11 The three given st lines are supposed to be of infinite length The locus of the centres of ○^{s} touching any pair must be the internal and external bisectors of the angle between them Four *different* centres will be given by the intersection of these loci, corresponding to what are known as the *inscribed* and *escribed* ○^{s} of the △ formed by the three given lines [See Probs 26, 27, p 194]

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1 On the given base describe a segment containing an angle equal to the given ∠ The pt or pts in which the arc of the segment cuts the given st line will give the required vertex

2 The required vertex is the intersection of the arc of the segment described on the base to contain an ∠ equal to the given ∠, and

(i) the circle, whose centre is an extremity of the base and radius is equal to the given side

(ii) the st. line parallel to the base at a distance from it equal to the given altitude

(iii) the circle whose centre is the middle pt of the base and radius equal to the given median

(iv) the perp to the base drawn through the given point

3. Because arc AP=arc BP, ∠ACP=∠BCP [*Th* 43]

4. Because $\angle ACB = K$, and $\angle AXB = \frac{1}{2}K$,

∴ $\angle CBX = \frac{1}{2}K$ [*Th* 16] $= \angle CXB$

∴ $CB = CX$ [*Th* 6], $AC + CB = AX$ = given length

5 On AB, the given base, describe a segment containing an $\angle$ equal to the given $\angle K$; also another segment containing an angle equal to $90° + \frac{1}{2}K$. From centre A, with radius equal to the given difference of the sides, describe a $\odot$ cutting the last drawn segment in X Join AX and produce it to cut the first segment in C Then ABC is the required triangle

Because $\angle AXB = 90° + \frac{1}{2}K$, $\angle CXB = 90° - \frac{1}{2}K$ [*Th* 1]

And $\angle XCB = K$, $\angle CBX = 90° - \frac{1}{2}K$ [*Th* 16]

∴ $CB = CX$ [*Th.* 6] $AC - CB = AX$ = given length

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1. Take any point A on the $\odot$ and draw RAQ perp to the radius OA, and ∴ a tangent, draw lines AB, AC making the $\angle^s$ RAB, QAC each 60°, let them meet the $\odot$ at B and C, then ABC is the triangle required [*Prob* 28]

To circumscribe an equil $\triangle$, draw tangents at B and C meeting at P and cutting RQ at R and Q respectively. Then PQR will be the triangle required

For $QA = QC$ [*Th* 47, Cor]; $\angle QCA = \angle QAC = 60°$.

∴ $\angle Q = 60°$ [*Th* 16]; similarly $\angle R = 60°$

∴ $\angle P = 60°$ [*Th.* 16], and the $\triangle$ PQR is equilateral

2 The circles are drawn as in Prob 25, 26, 27

In the case of an equilateral $\triangle$ the bisector of an angle also bisects the opposite side at rt $\angle^s$, ∴ the centres of the circum-circle and in-circle coincide at the intersection of the medians Let I be the common centre, ABC the $\triangle$, X the mid-pt of BC, I_1 the centre of the $\odot$ escribed to BC Then $AIXI_1$ is a st line [*Prob* 27 Note 2] Also $AI = 2XI$ [p 97, III Cor] ∴ the radius of the circum-circle is double that of the in-circle

Again, $\angle I_1BX = 60° = \angle ABC$; $\angle BXI_1 = 90° = \angle BXA$;

∴ by Theor 17, $XI_1 = XA = 3XI$ [p 97]

That is, the radius of the escribed $\odot$ is three times the radius of the inscribed $\odot$

3 The $\triangle^s$ being drawn, the $\bigcirc^s$ can be described as in Prob 25 The angle A will be found to be 64° in Cases (i), (ii), and 180° − 64° in Case (iii) [*Th* 16]

Since the side a is of the same length in each case, the $\triangle^s$ can be described on the same line as base, if A_1, A_2, A_3, be the vertices of the 3 $\triangle^s$, A_1 and A_2 will lie on the segment of a $\bigcirc$ containing an $\angle$ of 64° described on BC as chord, and A_3 on the conjugate segment of the same $\bigcirc$ [*Th* 39 and 40, *Converse*] the radii of the 3 circum-circles are equal

4. The $\triangle^s$ being described as in Ex 1, let O be the centre of the given $\bigcirc$, and X the mid-pt of BC

Then OA=4 cm OX=2 cm [p 97, III *Cor*]

But BXO is a rt $\angle^d$ $\triangle$ $BX^2 = OB^2 - OX^2$,

whence $BX = 2\sqrt{3}$ cm = 3 464 cm and CB=6 928 cm

Now area of $\triangle ABC = \frac{1}{2}$ AX BC $= \frac{1}{2}(6 \times 6\ 928)$ sq cm

$= 20\ 78$ sq cm

It was proved in Ex 1 that AQC was an equil $\triangle$ So for ARB, PBC each of these three $\triangle^s$ is equal to ABC in area $\triangle$ PQR is 4 times $\triangle$ ABC

5 With the Figure of Prob 26 we have

$$\text{area of } \triangle \text{IBC} = \tfrac{1}{2}\ \text{ID}\ \ \text{BC} \qquad [Th\ 25]$$

$$= \tfrac{1}{2}r\ a$$

Similarly $\triangle \text{ICA} = \frac{1}{2}r\ b$, and $\triangle \text{IAB} = \frac{1}{2}r\ c$

Now $\triangle \text{ABC} = \triangle \text{IBC} + \triangle \text{ICA} + \triangle \text{IAB}$

$$= \tfrac{1}{2}r\,(a+b+c)$$

In the numerical example r will be found to be 2 24 cm

6 With the Figure of Prob 27 we have, as above,

area of $\triangle\ I_1BC = \frac{1}{2}r_1a$, similarly for the $\triangle^s$ I_1CA, I_1AB,

Now $\triangle \text{ABC} = \triangle I_1CA + \triangle I_1AB - \triangle I_1BC$

$$= \tfrac{1}{2}r_1(b+c-a)$$

In the numerical example r_1 will be found to be 6 cm

7 By measurement R will be found to be 3 2 cm nearly, and p_1, p_2, p_3 to be 2 4 cm, 5 04 cm and 2 96 cm respectively

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1. Join the extremities of any two perp diameters Proof by Theorems 4 and 41

If r is the radius of the circle, and a the side of the square, we have $a^2=r^2+r^2$ [*Th* 29]$=2\ 25+2\ 25=4\ 50$ $a=2\ 12''$

2 Take two diameters at right angles, and draw the perps at their extremities Proof by Theors 14, 20, and 46

If r is the radius of the circle, the side of the square is $2r$,

$$\text{the area}=(2r)^2=4r^2$$

As in Ex 1, the area of the inscribed square is $2r^2$. area of the circumscribed square is twice that of the inscribed square

3 Draw the diagonals, which are axes of symmetry of the square [Ex 2, p 60] Hence the centre of the inscribed circle must lie on each of them [Ex 6, p 177], and be at their pt of intersection Also, by symmetry, the pts of contact of the ○ must be the middle pts of the sides, the radius is equal to half the side of the square

4. Let ABCD be the square, and O the intersection of its diagonals Describe a ○ with O as centre and OA as radius By measurement, the diameter will be found to be 8 5 cm

The diameter of the circle is a diagonal of the square; if d be its length, $d^2=6^2+6^2=72$, whence $d=8\ 48$ cm

5. For Construction, see Prob 10, p 83 Let a be the length of the other side, d of the diameter, then $d^2=a^2+9$ [*Th* 29], and $d=3\ 6$, whence $a^2=3\ 96$ And $a=2\ 0''$, to the nearest tenth of an inch

If PQRS is any inscribed rectangle, PR will be a diameter, for PQR is a rt angle O, the intersection of the diagonals, is the centre of the ○. Now the area of the rect = twice that of $\triangle$ PQR = PR $\times p$, where p is the perp from Q on PR

Hence the area is greatest when p is greatest, *i e* when $p=$ the radius Hence OQ must be perp to PR, by means of Theor 4, the rectangle must be a square

6 Let ABC be the triangle, I the centre, X the mid-pt of BC

As in Ex 2, p 198, AI=2XI $\quad$ XI$=\frac{r}{2}$

But BI^2=BX^2+XI^2, $\quad r^2=\left(\frac{b}{2}\right)^2+\left(\frac{r}{2}\right)^2$, whence $b^2=3r^2$

And for the square, since its diagonal is a diameter of the circle, $(2r)^2=a^2+a^2$, whence $a^2=2r^2$

$$3a^2=6r^2=2b^2$$

7 The angle subtended by a side of an inscribed square at any point on the major arc is half the angle subtended at the centre, that is, half a right angle

But the sum of the angles in the major and minor arc is two rt angles [*Th* 40],

hence the angle in the minor arc is $\frac{3}{2}$ of a rt angle.

8 Draw any two diameters AOC, BOD, and draw PQ, QR, RS, SP perp to the radii at A, B, C, D respectively, then these lines are tangents [*Th* 47]

Now, by Theor 13, PQRS is a par^m, also OP, OR bisect the vertically opposite angles AOD, BOC, respectively [*Th* 47], POR is a st line,

$\angle$PRS=$\angle$RPQ [*Th* 14]=$\angle$SPR [*Th* 47]

SP=SR [*Th.* 5] Similarly SP=PQ, PQ=QR

the par^m is equilateral, and is consequently a rhombus

9 In BC, CD, DA make BY, CZ, DW each equal to AX

Join XY, YZ, ZW, WX Then XYZW is the sq required

For the $\triangle^s$ XBY, YCZ, ZDW, WAX are all identically equal [*Th* 4], the fig XYZW is equilateral

Also, $\angle$ZYC=$\angle$YXB,

$\angle$ZYC+$\angle$XYB=$\angle$YXB+$\angle$XYB=one rt $\angle$ [*Th* 16]

$\angle$XYZ is a rt $\angle$ Similarly each of the other $\angle^s$ of the figure is a rt $\angle$ it is a square

10 The sq of minimum area is that obtained by joining in order the middle points of the sides of the given square

For a square is a minimum when its diagonal is a minimum and the least line that can be drawn between two opp sides of the given square is perp to those sides, this is obtained by joining the middle points

11 (i) The diags of a rect being equal, the intersection of the diagonals is the centre [Th 21 Cor 3]

(ii) Through the vertices of the given rect, draw lines, external to the figure, making ∠s of 45° with its sides

12 (i) Let OAB be the quadrant, AB being the arc

Bisect the rt ∠AOB by OD, meeting the arc at D, draw DF perp to OA, bisect the ∠ODF by DE, meeting OA at E, and at E draw EC perp to OA, meeting OD in C Then C is the centre of the required ⊙

For ∠CED = alt ∠EDF = ∠EDC [*Constr*]

CD = CE And if CG is drawn perp to OB, then CE = CG [Ex 2, p 49]

Finally, since the ∠s at E and G are rt angles, a ⊙ described from centre C with radius CD touches the arc and the radii of the quadrant

(ii) In this example it is understood that one angle of the square is to coincide with the angle between the radii

Bisect the ∠AOB by OD, and draw DF, DH perp to OA, OB

Then OFDH is the square required

For ∠FOD = ½ rt ∠, and ∠DFO is a rt ∠

∠ODF = ½ rt ∠, OF = DF And since the fig is a rectangular par^{m}, it is a square

(If two vertices of the sq are to lie on the arc, and the other two on the radii, see Ex 3, p 284)

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1 (i) Let O be the centre of the circle, OA a radius With centre A, radius AO, cut the O^{ce} at B Join AB Then △OAB is equilateral Set off chords equal to AB round the O^{ce} The resulting figure will be the hexagon required, for ∠AOB = 60°, or *one-sixth* of 360°

(ii) Let O be centre and OA a radius as before, draw OB at right angles to OA Bisect the ∠AOB by OC meeting the O^{ce} at C Set off chords equal to AC round the circle The resulting figure will be the octagon required, for ∠AOC = 45°, or *one-eighth* of 360°

(iii) As in (i) describe the equilateral triangle OAB Bisect ∠AOB by the line OC meeting the O^{ce} at C Set off chords equal to AC round the circle The resulting figure will be the dodecagon required, for ∠AOC = 30° = $\frac{1}{12}$ of 360°

2 (i) As in Ex 1 (i) inscribe a regular hexagon in the given ○, and draw the tangents at its angular points Taking the figure of Prob 30 let the tangents at A and B meet in T, those at B and C in R, and those at C and D in S

Now $\angle^s$ AOB, BOC are equal [*Th* 43], and OT, OR are their bisectors [*Th* 47, *Cor*], $\angle$TOB$=\angle$ROB

Also $\angle$TBO$=$rt $\angle=\angle$RBO

Hence $\triangle^s$ OBT, OBR are congruent [*Th* 17], TB$=$BR, that is RT$=$2RB

Similarly RS$=$2RC

But RB, RC are equal [*Th* 47, *Cor*] RT$=$RS

Similarly every pair of consecutive sides of the circumscribed hexagon are equal, and the figure is equilateral

Now, from quad^l AOBT, since the $\angle^s$ B, A are rt $\angle^s$,

$\angle$ATB$=$supplement of $\angle$AOB, [*Th* 16]
$=$supplement of $\angle$BOC, [*Th* 43]
$=\angle$BRC

Similarly every pair of consecutive angles of the figure are equal the circumscribed hexagon is regular

(ii) Inscribe the regular octagon as in Ex 1 (ii), and draw the tangents at its vertices The proof follows exactly as in Ex 2 (i)

3 (i) Let ABCDEF be the inscribed hexagon, and O the centre of the ○ Then because AO$=$OB and $\angle$AOB$=60°$, the $\triangle$AOB is equilateral [*Th* 16] and AB$=$OB each side of hexagon $=$radius of the ○

Hence $\triangle^s$ FAB, FOB are congruent [*Th* 7] and equal in area

Similarly $\triangle$BCD$=\triangle$BOD, and $\triangle$FED$=\triangle$FOD

, by addition, $\triangle$BFD$=$half the area of the hexagon

(ii) Since sum of $\angle^s$ AOB, BOC, COD$=180°$, AOD is a st line

Also ABOF is a rhombus and AO bisects BF at rt $\angle^s$
[*Ex* 4, p 59]

Let AO, BF meet in K

Then BK$=\frac{1}{2}$BF$=\frac{1}{2}a$, OK$=\frac{1}{2}$OA$=\frac{1}{2}b$

But BK$^2+$OK$^2=$OB^2 [*Th* 29], or, $\left(\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2=b^2$,

whence $a^2=3b^2$

4. This can only be done approximately Proceed as in Prob 30, making $\angle AOB = 51\frac{3}{7}°$ (*i e* $51\frac{1}{2}°$, to the nearest half degree)

$$\text{Then } \angle OBA = \tfrac{1}{2}(\text{supp}^{t} \text{ of } 51\tfrac{3}{7}°) = 64\tfrac{2}{7}°$$
$$\angle ABC = 2\angle OBA = 128\tfrac{4}{7}°$$

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1. Draw any st line $AB = 2\ 0''$ On AB describe an equilateral $\triangle AOB$ With centre O, radius OA, describe a ○, and in it set off chords BC, CD, DE, EF, FA each equal to AB Then ABCDEF is the req^d hexagon, and the ○ is its circumscribed ○ From O draw OP perp to AB Then OP is the radius of the in-○, for the circumcentre and in-centre of a *regular* polygon coincide [*Prob* 31]

Diameter of circum-circle $= 2OA = 4\ 0'$

Also $OP^2 = OA^2 - AP^2$ [*Th* 29] $= 2^2 - 1^2$, whence $OP = \sqrt{3}$ in

$\therefore$ diameter of in-circle $= 2\sqrt{3} = 3\ 46''$.

2 In the Fig of Prob 30, let the tangents at A and B cut at P Join OP, bisecting AB at rt angles at X [Ex 7 p 177]

It may be shewn that $\angle APX = 60°$, and $\angle PAX = 30°$;

and hence that $PX = \frac{1}{2}AP$, similarly $AP = \frac{1}{2}PO$

$$\therefore PO = 4PX, \text{ and } OX = 3PX$$

Now area of $\triangle OAB = \frac{1}{2}\ AB \cdot OX = \frac{1}{2}\ AB \times 3PX$

And fig $OAPB = \frac{1}{2}\ AB \cdot OP = \frac{1}{2}\ AB \times 4PX$

$$\therefore \triangle OAB = \tfrac{3}{4} \text{ fig } OAPB$$

$\therefore$, by addition, the in-hexagon $= \frac{3}{4}$ of circum-hexagon

In the numerical example, $OA = AB = 10$ cm, and $AX = 5$ cm

$$\text{Also } OX = \sqrt{OA^2 - AX^2} = \sqrt{75} = 5\sqrt{3} \text{ cm}$$

Now area of in-hexagon $= 6\triangle OAB = 3\ AB \cdot OX$

$= 150\sqrt{3}$ sq cm

$= 259\ 8$ sq cm

3 $$B+C+A=2A+2A+A=180°,$$
$$5A=180°, \quad A=36°$$

BC subtends an angle of 36° at the circumference, and one of 72° at the centre But the angle subtended at the centre by the side of an inscribed regular pentagon is $\frac{360°}{5}=72°$ BC is the side of a regular pentagon inscribed in the circle

4 (i) Proceed as in Ex 1

From O draw OP perp to AB Then OB=AB=2BP

$$\text{And } OP^2=OB^2-BP^2=4BP^2-BP^2=3BP^2$$

$OP=\sqrt{3}$ $BP=2\sqrt{3}$ cm

Now area of hexagon $=6\triangle OAB=3$ AB OP $=24\sqrt{3}$ sq cm
$=41\ 57$ sq cm

(ii) Let AB be the given line of 4 cm Bisect AB at rt angles by PQ, cutting AB at X

From XP cut off XC equal to XA From CP cut off CO equal to CA With centre O, radius OA, draw a ○, and step off chords each equal to AB round the ○ce

Join OA, OB, AC, and prove that $\angle ACX=45°$, hence from $\triangle$ COA, shew that $\angle COA=22\frac{1}{2}°$ central $\angle AOB=45°$

By measurement, OX=4 8 cm, approx

Now area of octagon $=8\triangle AOB=4$ AB OX
$=16\times 4\ 8$ sq cm
$=76\ 8$ sq cm, approx

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1

Circumference	Diameter	Value of π
16 0 cm	5 1 cm	3 137
8 8″	2 8″	3 143
13 5″	4 3″	3 140

$$\text{mean value}=\frac{9\ 420}{3}$$
$$=3\ 140$$

2 Circumference of cylinder $=\frac{75\ 4}{20}=3\ 77''$,

$$\pi=\frac{3\ 77}{1\ 2}=3\ 141, \text{ approx}$$

3 Circumference of wheel $=\frac{977}{400}=2\ 4425$ yds ,

$$\pi=\frac{2\ 4425\times 36}{28}=3\ 141, \text{ approx}$$

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1 (1) Circumference $=2\pi r=9\times\pi=9\times\frac{22}{7}=28\ 3$ cm

(2) Circumference $=2\pi r=200\times\pi=(200\times 3\ 1416)$ cm $=628\ 3$ cm

2 (1) Area $=\pi r^2=\pi\times(2\ 3)^2=(3\ 1416\times 5\ 29)$ sq in $=16\ 62$ sq in

(2) Area $=\pi r^2=\pi\times(10\ 6)^2=(112\ 36\times 3\ 1416)$ sq in $=352\ 99$ sq in

3 The radius of the circle will be half the side of the square, that is, 1 8 cm

circumference $=2\pi r=\pi\times 3\ 6=\frac{22}{7}\times 3\ 6=11\ 31$ cm

The area $=\pi r^2=\pi\times(1\ 8)^2=\frac{22}{7}\times 3\ 24=10\ 18$ sq cm

4. Let a be the side of the square, and r the radius of the O Then since the diagonal of the square is a diameter of the circle, we have $(2r)^2=a^2+a^2$, whence $a^2=2r^2$

Now area of square $=a^2=2r^2$ Area of circle $=\pi r^2$

the difference $=(\pi-2)r^2=1\ 1416\times 49=56$ sq cm

5 Req[d] area $=\pi\times(r_1{}^2-r_2{}^2)=\pi\times(r_1-r_2)(r_1+r_2)=\pi\times 1\ 4\times 10$ $=3\ 1416\times 14=43\ 98$ sq in

6 Let O be the common centre, P a pt on the outer circle, and PT the tangent from P to the inner circle

Then area of ring $=\pi\times(\mathrm{OP}^2-\mathrm{OT}^2)=\pi\times\mathrm{PT}^2$ [*Th* 29]

$=$ area of a circle whose radius is PT

7 The diagonals of the rectangle must be diameters of the circle, since they subtend right $\angle^s$ at the circumference Hence, if r be the radius of the circle, $(2r)^2=6^2+8^2=100$, whence $r=5$ cm

difference between the areas of circle and rect

$$=(\pi r^2-48) \text{ sq cm}$$
$$=(25\times 3\ 1416-48) \text{ sq cm}=30\ 5 \text{ sq cm}$$

8 Area of circle $=\pi\times 25$ sq in

side of square $=5\times\sqrt{3\ 14159}=5\times 1\ 77''=8\ 85''=8\ 9''$, to nearest tenth of an inch

9 Let r be the radius of the larger ○ in inches, and therefore $(r-1)$ that of the smaller

Then area of the ring $=\pi\{r^2-(r-1)^2\}$ sq in $=22$ sq in

$$\tfrac{22}{7}(2r-1)=22\text{, whence } r=4$$

the radii are $4\ 0''$ and $3\ 0''$ respectively

10 As in Ex 2, p 198, the radius of the circum-circle is double that of the in circle Denote them by $2r$, r respectively

Then difference between the areas of the two ○^{s}

$$=\pi\ 4r^2-\pi\ r^2=3\pi r^2$$

Now if ABC be the Δ, O the common centre of the two ○^{s}, and OD the perp from O to BC, we have $OB^2-OD^2=BD^2$,

or $4r^2-r^2=4$, whence $3r^2=4$

$\therefore$ diff between the two areas $=4\pi$ sq in

$$=12\ 57 \text{ sq in}$$

11 Distance between the two centres $=\sqrt{(1\ 5)^2+(0\ 8)^2}$ [*Th* 29]

$$=1\ 7''$$
$$=\text{sum of the two radii}$$

the ○^{s} touch externally [*Th* 48]

12 As in Ex 11, distance between the centres $=2\ 0''$, which is also the sum and difference of the two pairs of radii

the 1st and 2nd ○^{s} touch externally,

and the 1st and 3rd ○^{s} touch internally

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1. Let AB, CD be the two par^l lines cut by PQ
   Bisect the $\angle^s$ APQ, CQP by lines which meet at O Draw OX, OY, OZ perp to AB, CD, PQ
   Then the $\triangle^s$ XPO, ZPO are identically equal [*Th* 17]
   OX=OZ Similarly OZ=OY And since the $\angle^s$ at X, Y, Z are rt. angles, a $\bigcirc$ described from centre O with radius OX touches the given lines at X, Y, Z
   A second $\bigcirc$ is obtained by bisecting the $\angle^s$ BPQ, DQP
   Again since OX, OY are perp to par^l lines, they may be shewn to be in the same st line
   Hence the diam of each $\bigcirc$ is the perp distance between the given par^{ls}, the $\bigcirc^s$ are equal

2 Let ABC, DEF be the two $\triangle^s$ having BC=EF, and the $\angle$BAC =$\angle$EDF Let O, O′ be the centres of their circum-circles Join OB, OC, O′E, O′F.
   Then $\angle$BOC=2$\angle$BAC=2$\angle$EDF=$\angle$EO′F. [*Th* 38]
   the *isosceles* $\triangle^s$ BOC, EO′F have their vertical $\angle^s$ equal, their remaining angles are equal [*Th* 16]; and BC=EF, the $\triangle^s$ are congruent [*Th* 17], and the radii OB, O′E are equal

3 Join BS, CS

Then since	SA=SB,	$\angle$SAB=$\angle$SBA
And since	SA=SC,	$\angle$SAC=$\angle$SCA
But since I is in AS,		$\angle$SAB=$\angle$SAC
		$\angle$SBA=$\angle$SCA

   Hence the $\triangle^s$ BAS, CAS may be shewn to be congruent [*Th* 17]
   AB=AC

4. Let ABC be the $\triangle$, rt-angled at C, and let the inscribed $\bigcirc$ touch the sides BC, CA, AB at D, E, F, and let I be its centre
   Join ID, IE Then clearly the fig IC is a square
   Hence the diam of inscribed $\bigcirc$=DC+CE
   And since BCA is a rt angle, the diam of the circum $\bigcirc$=BA [*Th* 41]
   =BF+AF=BD+AE [*Th* 47 *Cor*]
   the sum of the diams =DC+CE+BD+AE=BC+AC,

5 D is to be opp to A, E to B, and F to C
In the $\triangle$AFE, AF=AE [*Th* 47, *Cor*]

$$\angle AFE = \angle AEF = 90^\circ - \frac{A}{2} \quad [Th\ 16]$$

But $\angle$AFE=$\angle$EDF [*Th* 49] $\angle EDF = 90^\circ - \frac{A}{2}$

6 Join BI, BI_1, also CI, CI_1

Then BI, BI_1 are respectively the internal and external bisectors of the $\angle$ABC,

the $\angle IBI_1$ is a rt angle [Ex 6, p 13]

Similarly the $\angle ICI_1$ is a rt angle

the four points I, B, I_1, C are concyclic [*Th* 41]

7 Take the Figure of p 194

Then since AF=AE [*Th* 47, *Cor*], the diff of AC and AB=the diff of EC and FB, that is the diff of CD and DB

8 [In the Figure taken AB is greater than AC]

Since the $\angle$ASB is twice the $\angle$ACB, [*Th* 38]

$$\angle SAB = 90^\circ - C, \quad [Th\ 16]$$

$$\begin{aligned}\angle SAI &= \angle IAB - \angle SAB \\ &= \tfrac{1}{2}A - (90^\circ - C) \\ &= \tfrac{1}{2}A - (\tfrac{1}{2}A + \tfrac{1}{2}B + \tfrac{1}{2}C) + C \\ &= \tfrac{1}{2}C - \tfrac{1}{2}B\end{aligned}$$

Again, as above, $\angle SAB = 90^\circ - C$,

and $\angle DAB = 90^\circ - B$,

$$\begin{aligned}\angle DAS &= \angle DAB - \angle SAB \\ &= C - B = 2\angle SAI\end{aligned}$$

9 Let P, Q, R, S be the centres of the $\odot^s$ about the $\triangle^s$ in the order named

Then clearly SR bisects DO at rt angles [*Prob* 25]

Similarly PQ bisects OB at rt angles

SR is par[l] to PQ Thus also SP is par[l] to RQ,

SPQR is a par[m]

10 Join BO, OC

Since $\angle BAO=\angle CAO$, arc BO=arc OC [*Th* 42],

chord BO=chord OC [*Th* 45]

Again the ext $\angle BIO=\angle IAB+\angle IBA=\frac{1}{2}A+\frac{1}{2}B$

Also the $\angle IBO=\angle OBC+\angle IBC=\angle OAC+\angle IBC$ [*Th* 39]

$=\frac{1}{2}A+\frac{1}{2}B$

$\therefore \angle BIO=\angle IBO$, $\therefore OI=OB=OC$

$\therefore$ O is the centre of the ○ about BIC

11 Let AB be the given base Draw PQ par[l] to AB at a distance from it equal to the given altitude

Describe ○s from centres A and B with radius equal to the given radius of circum ○, let these ○s intersect at O, on the same side of AB as PQ. From centre O with radius OA describe a ○ cutting PQ at C or C′ Then either of the △s ABC, ABC′ satisfies the required conditions

12 AB, BC, CA pass through F, D, E respectively. [*Th.* 48]

At F, E, D draw the common tangents FP, EQ, DR to the pairs of ○s which touch at those points

Then if these 3 lines are not concurrent, let FP, EQ, meet in O, and from O draw OK perp to BC

Then since $AO^2-OB^2=AF^2-FB^2$ [Ex. 2, p 123]

and $AO^2-OC^2=AE^2-EC^2$,

we have, by subtraction,

$$OC^2-OB^2=EC^2-FB^2,$$

$$=CD^2-DB^2 \quad [\textit{Th } 47, \textit{Cor}]$$

$$\text{But } OC^2-OB^2=CK^2-KB^2 \quad [\textit{Ex } 2, \text{p } 123]$$

$$\therefore CD^2-DB^2=CK^2-KB^2$$

$$\therefore (CD+DB)(CD-DB)=(CK+KB)(CK-KB)$$

$$\text{But } CD+DB=CK+KB;$$

$$\therefore CD-DB=CK-KB,$$

$\therefore$, by addition, $2CD=2CK$, whence D and K coincide

Thus the three tangents meet in O

$\therefore$ OD=OF, and OF=OE [*Th* 47, *Cor*],

$\therefore$ a ○ with centre O and radius OD passes through D, E, F and touches the sides of the △ABC at those points, for $\angle$s OFB, ODC, OEA are rt $\angle$s.

$\therefore$ the ○DEF is the in-○ of △ABC and the circum-○ of △DEF

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1 Let the perps from A and B on BC and AC meet those sides at D and E and intersect in O and let AD meet the circum circle in G Join BG

Then in the $\triangle^s$OEA, ODB,

because $\angle$OEA$=\angle$ODB, each being a rt angle,

and $\angle$EOA$=\angle$DOB, [*Th* 3]

remaining $\angle$EAO$=$remaining $\angle$DBO [*Th* 16]

But $\angle$CAG$=\angle$CBG, [*Th* 39]

$\angle$DBO$=\angle$DBG

Hence the $\triangle^s$ DBO, DBG are congruent [*Th* 17]

DO$=$DG

2 In the Fig of p 208, produce ED to X

It has been shewn that $\angle$EDC$=\angle$FDB [p 208 II *Cor*]

But $\angle$EDC$=\angle$BDX [*Th* 3], $\angle$FDB$=\angle$BDX

That is, the ext $\angle$FDX is bisected by BD and so on for the other $\angle^s$ of the pedal $\triangle$

The latter part of the proposition may be solved in a similar manner

3 With the Fig of p 208, since the $\angle^s$ AFO, AEO are rt angles (*hyp*), the four points A, F, O, E are concyclic

the $\angle^s$ FAE, FOE together $=$two rt angles [*Th.* 40]

That is, the $\angle^s$ BAC, BOC together $=$two rt angles [*Th* 3]

4 With the Fig of p 208, consider the $\triangle$OBC

Here BF is the perp from B on the opp side CO produced

and CE is the perp from C on the opp side BO produced

Now BF and CE intersect in A, and AO produced is perp to BC [p 207, I] Hence A is the orthocentre of the $\triangle$OBC

5 Consider the $\odot^s$ circumscribed about the $\triangle^s$ABC, OBC, and let X be any point on the $\odot^{ce}$ of the $\odot$BOC, on the side of BC remote from O

Then the $\angle^s$ BOC, BXC are supplementary [*Th* 40],

and the $\angle^s$ BOC, BAC are supplementary [*Ex* 3],

$\angle$BXC$=\angle$BAC

Hence the segments BAC, BXC stand on the same base, and contain equal angles, the circles of which these segments are parts are equal [*Ex* 2, p 206]

6 Consider the $\triangle$ FAB BD is perp to the side AF [*Th* 41], and AE is perp to BF for the same reason

G, their point of intersection, is the orthocentre of the $\triangle$ AFB

FG (produced, if necessary) is perp to AB [p 207, I]

7 Here $\angle$ BCK $=\angle$ BAK, in same segment

$=$ compᵗ of $\angle$ AKB [*Th* 11]

$=$ compᵗ of $\angle$ ACB [*Th* 39]

$=\angle$ OBC [*Th* 16]

Similarly $\angle$ KBC $=\angle$ BCO,

BO is parˡ to KC, and BK parˡ to OC [*Th* 13]

8. With the figure of the last exercise, since BOCK is a parᵐ, the diagonals bisect one another [*Th* 21, *Cor* iii] That is, KO passes through the middle point of BC Hence the st line joining O to the middle point of BC, passes through K

9 From Ex 8, we see that the st line joining the orthocentre to the middle point of the base passes through an extremity of the diamʳ drawn from A

$\angle$ APQ is a rt angle [*Th* 41], and since AP is also perp to BC, PQ is parˡ to BC [*Th* 13]

10 Let SX be the perp drawn from S the centre of the circum-○ on BC Then by Ex 8, AS and OX meet the ○ᶜᵉ at the same point Q. And SX, passing through the middle point of AQ, is parˡ to AO, SX is half of AO [*Ex* 1, 3, p 64]

11 Let S be the centre of the ○ circumscribed about the $\triangle$ ABC, and A′, B′, C′ the centres of the ○ˢ about the $\triangle$ˢ OBC, OCA, OAB

Then it follows from Ex 5 that BSCA′ is a rhombus, and SA′ and BC bisect one another at rt angles Also SB′ and AC

Hence, by Ex 10, AO $=$ A′S Similarly OB $=$ SB′

Again SA′ and AO are parˡ, for both are perp to BC

Similarly SB′ and BO are parˡ, $\angle$ AOB $=\angle$ A′SB′

A′B′ $=$ AB [*Th* 4] Similarly B′C′ $=$ BC and C′A′ $=$ CA

It may be noticed that in the $\triangle$ˢ ABC, A′B′C′ the orthocentre of each is the circumcentre of the other

12 Let A be the vertex, O the orthocentre, and S the centre of the circum-○

From centre S with radius SA describe a ○, this is the circumcircle of the required △

Join AO and produce it to meet the ○ce at G

Bisect OG at D, and draw the chord BC perp to AG Join AB, AC Then ABC shall be the required △ Proof follows from Ex 1

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1 Let BC be the given base, and BAC any △ of the system, having the vertical ∠BAC constant in magnitude, but not fixed in position Let the bisectors of the exterior angles at B and C intersect at I_1

Then $\angle I_1BC = \frac{1}{2}(180° - B) = 90° - \frac{B}{2}$

Similarly, $\angle I_1CB = 90° - \frac{C}{2}$

But in $\triangle I_1BC$,

$$\angle I_1 + \angle I_1BC + \angle I_1CB = 180° \quad [Th\ 16]$$

Hence $\angle I_1 = \frac{B}{2} + \frac{C}{2} = 90° - \frac{A}{2} =$ constant

since the base BC is fixed, the locus of I_1 is the arc of a segment of a circle [*Th* 39, *Converse*]

NOTE. The locus of I in IV p 210 and the locus of I_1 are conjugate arcs of the same ○

2 Let the bisectors meet at X

Then ∠PAB, QBA together = two rt angles [*Th* 14]

∠XAB, XBA together = one rt angle [*Hyp*]

∠AXB is a rt angle [*Th* 16]

And since AB is fixed, the locus of X is a circle on AB as diameter [Ex 1, p 165]

3 Let A be the fixed point, C the centre of the ○, and APQ any chord through A, meeting the ○ce at P, Q Let X be the middle point of PQ. Then CX is perp to PQ [*Th* 31]

That is, the ∠AXC is a rt angle, and since AC is a fixed base, the point X lies on the ○ce of a ○ on AC as diam

[*Ex* 1, p 165]

(i) If A is external, the locus is that part of the ○ on AC which is intercepted within the given ○.

(ii) If A is on the ○ce, the locus is a complete ○ described on the radius AC as diam, and having internal contact with the given ○

(iii) If A is internal, the locus is a complete ○ falling within the given ○

4 Let A be the given point, and B the common centre of the concentric ○s. Let P be the point of contact of a tangent from A to any one of these ○s Then APB is a rt angle [*Th* 46]

And since A and B are fixed points, the locus is a circle on AB as diam [*Ex* 1, p 165]

5 Let A, B be the fixed points on the ○ce, PQ the arc of constant length but variable position Let AP, BQ intersect at X. To find the locus of X [In the fig taken AP, BQ intersect when produced outside the ○] Join PB

Then $\angle APB = \angle AXB + \angle PBX$ [*Th* 16],

or $\angle X = \angle APB - \angle PBQ$.

But these are constant angles, being subtended by the constant arcs AB and PQ [*Th* 39], the $\angle X$ is constant

the locus is the arc of a segment described on AB [*Th* 39, *Conv*]

6 Let PA, QB intersect at X Join PB [In the fig taken PQ and AB do not intersect within the circle, and X is also external]

Then $\angle X$ is the diff of $\angle$s PBQ, XPB [*Th* 16].

But $\angle PBQ$ is constant, being a rt angle [*Th* 41]

Also $\angle XPB$ is constant, being subtended by the fixed arc AB

the $\angle X$ is constant, and since the points A, B are fixed, the locus of X is the arc of a segment [*Th* 39, *Conv*] When X is internal, the $\angle X$ is supplementary to the value found above, and the conjugate segment of the locus is obtained

7 It follows that AP=AC, $\angle APC = \angle ACP$

But $\angle BAC$ = sum of $\angle$s APC, ACP [*Th* 16]

$\angle BAC$ is double of $\angle APC$

Or, $\angle BPC$ is half of $\angle BAC$, and is therefore constant

Then, since BC is fixed, the locus of P is the arc of a segment on BC [*Th* 39, *Conv*]

8 The intersection of the diagonals is X, the middle point of BC [*Th* 21, *Cor* iii] Join X to D, the middle point of AB

Then DX is par[1] to AC [*Ex* 2, p 64]

∠DXB=∠ACB [*Th* 14]

But ∠ACB is constant [*Th* 39]

∠DXB is constant, and D, B are fixed points

the locus of X is the arc of a ○

9 Let A be the point of intersection of the rulers

Then PXQA is a rectangle

AX=PQ, which is constant, and the point A is fixed Hence the locus of X is the quadrant of a circle described from the centre A with radius PQ

10 [Take the figure in which PA and PB must both be produced to meet the second ○ce] Let AY, BX intersect at R

Then the locus of R is required

Now ∠ARB=sum of ∠s RBY, RYB [*Th* 16]

=sum of ∠s at P, X, Y [*Th* 16],

and these are all constant, being subtended by fixed arcs

∠ARB is constant, and since the points A and B are fixed, the locus is part of a circle If PA or PB cuts the ○ce without being produced, the ∠ARB=the supplement of the sum of the ∠s P, X, Y Hence the rest of the circle is obtained

11 Let PH and KQ intersect at X Required the locus of X

From the △PXQ it will be seen by Theor 16 that the ∠X =the diff of the ∠s HPA, AQK, both of which are constant, since they stand on the fixed arcs HA, AK

And since H, K are fixed points, the locus of X is part of a ○

If P and Q are on the same side of A, the value of the ∠X is supplementary to that found above, and the rest of the ○ is obtained

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1 Take the Figure of the text, and join PB

Then since the ∠s PFB, PDB are rt angles, the pts P, F, B, D are concyclic

$\angle$ PDF $=\angle$ PBF, in the segment PFBD
$=\angle$ PCA, in the segment PCBA
E, D, C, P are concyclic. [*Th* 39 *Converse*]
$\angle$ PEC $=\angle$ PDC, in the same segment,
$=$ a rt angle

2 With the same Figure, because E, P, F, A are concyclic,
$\angle$ EPA $=\angle$ EFA, in the same segment
Similarly $\angle$ EPC $=$ supp^t of $\angle$ EDC $=\angle$ EDB
, by addition, $\angle$ APC $=$ sum of $\angle^s$ EFB, EDB
$=$ supp^t of $\angle$ ABC [*Th* 16]
P lies on the O^{ce} of the O circumscribed about the $\triangle$ ABC [*Th* 40 *Converse*]

3 Draw PD, PD′, PE, PF perp respectively to the four lines BC, B′C′, ACC′, ABB′
Then since P is on the circum circle of the $\triangle$ ABC, the points E, F, D are collinear [V p 212]
And since P is on the circum-circle of the $\triangle$ AB′C′,
the points E, F, D′ are collinear
Hence D and D′ both lie on the st line through E and F

4 Let ABC be the $\triangle$, P the given point on the circumscribed $\odot$
Let PF, PD be the perps on AB, BC so that FD produced is the pedal of P Draw AH perp to BC, and produce it to meet the O^{ce} at G Take HO equal to HG Then O is the orthocentre [*Ex* 1, p 209]
Let OP meet the pedal of P at X Then shall OX $=$ XP
Draw PB, PC Let PG, produced if necessary, meet the pedal at K, and BC (or BC produced) at L Join OL
[The proof given below is for an acute-angled triangle P is taken in the arc BG F falls within AB, and PG meets BC produced]
Then $\angle$ PDK $=\angle$ PBF [*Ex* 5, p 163] $=$ supp^t of $\angle$ ACP [*Th* 40]
$=$ supp^t of $\angle$ AGP [*Th* 39] $=\angle$ DPG [*Th* 14]
So that $\angle$ KDL $=\angle$ KLD, since $\triangle$ PDL is rt angled [*Th* 16]
PK $=$ KD $=$ KL
But by Theor 4, the $\triangle^s$ HLG, HLO are congruent
$\angle$ OLH $=\angle$ DLK $=\angle$ KDL, XK, OL are par^l But K is the middle point of PL, X is the middle point of OP
[*Ex* 1, p 64]

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(i) $AE=AF$, and $BD=BF$, and $CD=CE$ [*Th* 47, *Cor*],

$$AE+BD+DC=s,$$

or $\quad AE+a=s, \quad AE=s-a$

Similarly $\quad BD=s-b$, and $CD=s-c$

(ii) $\quad AE_1=AF_1 \quad$ [*Th* 47, *Cor*]

And
$$\begin{aligned} AE_1+AF_1 &= AC+CE_1+AB+BF_1 \\ &= AC+CD_1+AB+BD_1 \\ &= AC+BC+AB=2s, \end{aligned}$$

$$AE_1=AF_1=s$$

(iii) $\quad CD_1=CE_1=AE_1-AC=s-b$ [by (ii)],

$BD_1=BF_1=AF_1-AB=s-c$

(iv) $\quad CD=BD_1$, for each $=s-c$ [by (i) and (iii)],

$BD=CD_1$, for each $=s-b$

(v) $\quad EE_1=AE_1-AE=s-(s-a)=a$

Similarly $\quad FF_1=a$

(vi)
$$\begin{aligned} \triangle ABC &= \text{the } \triangle BIC+\triangle CIA+\triangle AIB \\ &= \tfrac{1}{2}ra+\tfrac{1}{2}rb+\tfrac{1}{2}rc \\ &= \tfrac{1}{2}r(a+b+c) \\ &= rs \end{aligned}$$

Again,
$$\begin{aligned} \triangle ABC &= \triangle ABI_1+\triangle ACI_1-\triangle BCI_1 \\ &= \tfrac{1}{2}r_1c+\tfrac{1}{2}r_1b-\tfrac{1}{2}r_1a \\ &= \tfrac{1}{2}r_1(c+b-a) \\ &= r_1(s-a) \end{aligned}$$

(vii) See the Fig to p 213, and suppose the $\angle$ at C to be a rt angle Then the fig IDCE is a square, and

$$r=ID=CE=s-c \text{ [by (i)]}$$

Again, if the $\angle$ at C is a rt angle, the fig $I_1D_1CE_1$ is a square, so that $r_1=I_1D_1=CE_1=s-b$

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(i) The points A, I, I_1, are collinear, for I and I_1 both lie on the bisector of the vert $\angle BAC$ [*Prob* 27, Note 2]

Similarly I, I_2 and I, I_3 are on the bisectors of the $\angle$s ABC, ACB

(ii) For AI_2 and AI_3 being the bisectors of opp vert ∠s [*Prob* 27] are in the same st line. So for the two other groups.

(iii) The ∠CAI_2=∠BAI_3, being halves of opp vert ∠s, and ∠CAI=∠BAI [*Const*]

∴ I_1IA is perp to I_2I_3; similarly I_2IB is perp to I_3I_1, and I_3IC to I_1I_2.

∴ I is the orthocentre of the △ $I_1I_2I_3$ and ABC is the pedal △ of the △ $I_1I_2I_3$.

Hence the △s BI_1C, CI_2A, AI_3B are equiangular.
[II *Cor* ii, p 208]

(iv) The chord of contact of the tangents to inscribed ○ from A is perp to AI [*Ex* 7, p 177], and is therefore par^l to I_2I_3. So for the other sides. Hence the △s are equiangular.

(v) and (vi) follow from the fact that I is the orthocentre of the △ $I_1I_2I_3$ [*Exs* 4, 5, p 209]

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1 (i) $DD_2 = BD_2 - BD = s-(s-b)=b$, [(i), (ii), p 213]
$D_1D_3 = CD_3 - CD_1 = s-(s-b)=b$

(ii) Similarly $DD_3=D_1D_2=c$

(iii) $D_2D_3=BD_2+BD_3=s+(s-a)=b+c$

(iv) $DD_1 = BD - BD_1=(s-b)-(s-c)=c-b$

2 Follows directly from II p 208 and Ex 2, p 209

3 Take the figure of p 214

Since BI, BI_1 are respectively the internal and external bisectors of the ∠ABC, ∴ ∠IBI_1 is a rt angle.

So also the ∠ICI_1 is a rt angle. ∴ the points I, B, I_1, C are concyclic.

∴ ∠BI_1I=∠BCI=½C, and ∠CI_1I=∠CBI=½B [*Th* 39]

Hence by addition the ∠BI_1C=½B+½C
= comp^t of ½A [*Th* 16]

That is, the ∠BI_1C is constant; and the base BC is fixed, ∴ the locus is the arc of a segment.

4. Given the base BC and the vert angle, the vertex A must lie on the arc of a certain segment described on BC as base That is, the three points A, B, C lie, for all positions of A, on a fixed circle, for if an arc of a circle is fixed, the whole circle is fixed [*Th* 32] the centre is fixed

5. Take the figure of p 214 Required the locus of I_2

Since the $\angle^s$ ICI_2, IAI_2 are rt angles,

the points I, C, I_2, A are concyclic

$\angle II_2C = \angle IAC = \frac{1}{2}A$ [*Th* 39]

Hence the locus of I_2 is the arc of a segment on BC as base, capable of containing an angle equal to $\frac{1}{2}A$

6 On the base BC describe a segment containing *one rt angle* + $\frac{1}{2}$ *the given angle* X, then the centre of the inscribed ○ is on this arc [IV, p 210]

At D, the given point in BC, draw a line perp to BC cutting the arc at I Then I must be the centre of the inscribed ○, and ID is its radius

From centre I, with radius ID, draw the inscribed ○, to which draw tangents from B and C If these tangents produced meet at A, then ABC is the required triangle

Prove by a method converse to IV, p 210 that the

$\angle BAC$ = the given angle X

7 On BC the given base describe a segment containing *one rt angle* $-\frac{1}{2}$ *the given angle* [*Ex* 3], then the centre of the escribed ○ must be on this arc From this point proceed as in the last Example

8 Now I and I_1 lie on the bisector of the $\angle BAC$

Let AII_1 cut the circum-○ at P Join PB

Then $\angle PBI = \angle PBC + \angle CBI = \angle PAC + \angle CBI$ [*Th* 39]

$= \frac{1}{2}A + \frac{1}{2}B$

Also ext $\angle PIB = \angle IAB + \angle IBA = \frac{1}{2}A + \frac{1}{2}B$

$\angle PBI = \angle PIB$, $PI = PB$

And $\angle I_1BI$ is a rt angle

$\angle PBI_1 = \angle PI_1B$ [*Th* 16] $PB = PI_1$

Hence P is the middle point of II_1

9 Take the Figure of p 214

Since the $\angle$s I_1BI_3, I_2CI_3 are rt angles [(v), p 214],

∴ a ○ on I_2I_3 as diam passes through B and C

Let the circum-○ cut I_2I_3 at Q (and let Q be in AI_3) Join QB

Then $\angle QI_3B = \angle II_3B + \angle II_3A = \angle IAB + \angle IBA$ [*Th* 39]

$= \frac{1}{2}A + \frac{1}{2}B$

And $\angle I_3QB = \angle C$ [Ex 5, p 163],

∴ from $\triangle I_3QB$, the $\angle I_3BQ = \frac{1}{2}A + \frac{1}{2}B$ [*Th* 16]

∴ $QI_3 = QB$ Similarly $QI_2 = QB$

∴ Q is the centre of the ○ through I_2, C, B, I_3

[NOTE Observe that the points P and Q in Exs 8, 9 are the extremities of the diam of the circum-○ perp to BC]

10. In the $\triangle$ formed by joining the three points A, B, C inscribe a circle and let the points of contact be D, E, F (D being opp to A, &c)

Then AE=AF, BD=BF, and CE=CD [*Th* 47, *Cor*]

Hence ○s described from the centres A, B, C, with radii AF, BD, CE will clearly satisfy the given conditions There will be *four* solutions in all, for solutions may also be obtained from the three *escribed* ○s

11 12 It has been proved in (v), p 214, that if I, I_1, I_2, I_3 are the centres of the inscribed and escribed circles of the $\triangle$ ABC, each of these four points is the orthocentre of the triangle formed by the other three, and that the original $\triangle$ ABC is the pedal triangle

Hence given any three of the points I, I_1, I_2, I_3, we have only to draw the pedal triangle of the triangle so formed

13 Let AX, AY determine the given vert $\angle$ Mark off AE_1, AF_1 each equal to half the given perimeter and describe a circle to touch AX, AY at E_1 and F_1 This is an escribed ○ of the required triangle

As in Ex 6, p 189 describe a ○, with radius equal to the given radius, to touch AX, AY This is the in-circle

Then if either of the transverse common tangents of these two ○s be drawn, meeting AX, AY in B and C, the $\triangle$ ABC will satisfy the required conditions

14. Draw the inscribed ○ as in the last Example, and from the given vertex as centre, with the given altitude as radius, describe a circle Then draw either of the direct common tangents to the two ○s

15 Take the figure of p 214

Since BI and BI_1 are the internal and external bisectors of the ∠ABC, the ∠IBI_1 is a rt angle

Similarly, the ∠ICI_1 is a rt angle II_1 is the diameter of the ○ about IBI_1C But the $○^{ce}$ of the ○ about the △ABC bisects II_1 [Ex 8, p 215] that is, the centre of the ○ about BIC lies on the $○^{ce}$ of the ○ about ABC

Page 218.

1 Take the figure of p 218

If the base and vert ∠ are given, then the circum ○ is fixed in position and magnitude [*Th* 39], hence the radius of the nine points ○ (being half that of the circum-○) is given that is, XN is constant But X is a fixed point, the locus of N is a ○, of which X is the centre

2 For by Ex 4, p 209, each of the △^{s} ABC, AOB, BOC, COA have the same pedal triangle, and therefore the same nine points ○, for the nine-points ○ circumscribes the pedal triangle

3 For by Ex (x), p 211, the △ABC is the pedal triangle of each of the four triangles formed by joining three of the points I, I_1, I_2, I_3

4 For, in the figure of p 218, both O and S are fixed N, the middle point of SO is fixed

And since the circum ○ is given, the radius of the nine-points ○ is given [(ii), p 217] Hence the nine-points ○ is fully determined

5 Take the figure of p 208

Let BC be the fixed base of the △ABC, having its vert ∠BAC constant in magnitude Then the circum-○ is fixed in position and magnitude [Ex 4, p 215] Hence the ○ about the pedal △DEF is fixed in magnitude, for its radius is half that of the circum-○ [p 217]

Now the ∠FDE at the $○^{ce}$ is constant, for it is the supp^t of twice the vert ∠A [II, p 208, *Cor*] the chord FE is of constant length [*Th* 42, 45]

6 Take the Fig of p 214

Now the base BC is given, and the vert ∠BAC is constant, the circum-O is fixed in magnitude and position

But ABC is the pedal △ of the △ $I_1I_2I_3$ [(v), p 214],

the circum-O of the △ABC is the nine-points O of $I_1I_2I_3$

Hence if AI_1, and I_2I_3 cut the O about ABC at X and Y, these are the middle points respectively of II_1, and I_2I_3 [VIII, p 216]

But since XAY is a rt ∠ [Ex 6, p 13], XY is a diam, and its middle point N is the centre of the O about ABC, and is a fixed point

But X is a fixed point (the middle point of the arc BC, since ∠BAX=∠CAX), Y is a fixed point

Draw YS perp to I_2I_3 meeting IN produced at S

Then S is the centre of the O about $I_1I_2I_3$, for this centre must lie both in YS and IN produced [p 217]

And SN=NI also YN=XN (proved), and ∠SNY=vert opp ∠XNI,

SY=IX [*Th* 4]=$\frac{1}{2}$ II_1 (proved above)

But II_1 is a diam of the O about the △BIC, and this is a fixed O, for the base and vert ∠ are constant [IV, p 210] Hence SY is constant, and as Y has been shewn to be a fixed point, the locus of S is a O about Y as centre

PART IV

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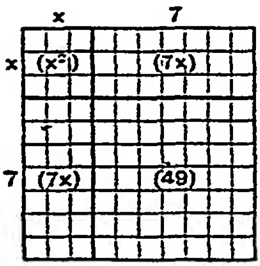
2



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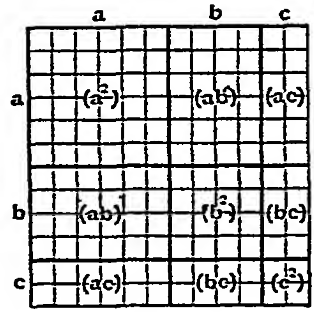

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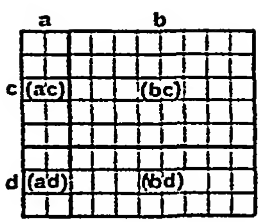

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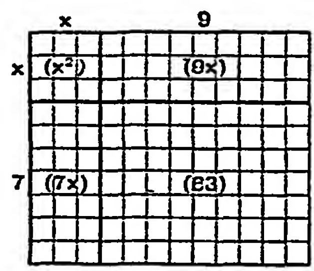

(iv)

3 Since $AB = 4$ cm, the square $AC = 16$ sq cm

$\therefore$ fig XC = fig AC − fig AE

$= (16 - 9{\cdot}6)$ sq cm $= 6{\cdot}4$ sq cm

4. The rect $AX \cdot XB$ = fig $XC = 3{\cdot}36$ sq in

Also $AX = 2{\cdot}1''$, $\quad XB = \dfrac{3{\cdot}36}{2{\cdot}1} = 1{\cdot}6''$

$AB = AX + XB = 3{\cdot}7''$

5. AX^2 = fig $AG = 36$ sq cm, $\quad AX = 6$ cm

But the rect $AX \cdot XB = 24$ sq cm, $\quad XB = \dfrac{24}{6} = 4$ cm

$AB = AX + XB = 10$ cm

6 AX^2 = fig $AG = 9{\cdot}61$ sq in; $\quad AX = \sqrt{9{\cdot}61} = 3{\cdot}1''$

But the rect $AB \cdot AX$ = fig AF = fig AG − fig $DG = 3{\cdot}1$ sq in

$AB = \dfrac{3{\cdot}1}{3{\cdot}1} = 1{\cdot}0''$

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1. Since $a^2+b^2=21^2+17^2=730$ sq cm, and $c^2=10^2=100$ sq cm, the deficit $=630$ sq cm

But, by Theor 55, this deficit measures twice the rect contained by BC and the projection of AC on BC

$$\text{projection of AC on BC}=\frac{630}{2BC}=\frac{630}{42}=15 \text{ cm}$$

2 Since the $\angle C$ is acute, $AB^2=BC^2+CA^2-2\ CA\ CE$ [*Th* 55]

But $AB^2=AC^2$, $\quad BC^2=2CA\ CE$

3 Draw AD perp to BC With centre A, and radius AC, cut BC, or BC produced, at E Join AE Then in both cases

$$\angle AEC=\angle ACE=60^\circ \qquad \angle EAC=60^\circ \quad [\textit{Th } 16]$$

Hence the $\triangle ACE$ is equilateral, and it can be shewn that $CD=DE$, from the congruent $\triangle^s$ ADC, ADE [*Th* 18]

$$CD=\tfrac{1}{2}CE=\tfrac{1}{2}CA$$

(i) Here $AB^2=BC^2+CA^2-2BC\ CD$ [*Th* 55]

$=BC^2+CA^2-BC\ CA$, from the above

or $c^2=a^2+b^2-ab$

(ii) Here $AB^2=BC^2+CA^2+2BC\ CD$, [*Th* 54]

$=BC^2+CA^2+BC\ CA$,

or $c^2=a^2+b^2+ab$

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Take the Fig of Theor 56

Then from the $\triangle AXB$, $AB^2=BX^2+AX^2+2BX\ XD$ [*Th* 54]

And from the $\triangle AXC$, $AC^2=XC^2+AX^2-2CX\ XD$ [*Th* 55]

by subtraction, remembering that $BX=XC$, we have

$$\begin{aligned} AB^2-AC^2&=2BX\ XD+2CX\ XD \\ &=2XD(BX+CX) \\ &=2XD\ BC \end{aligned}$$

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1 $\quad AB^2=AB\ AX+AB\ XB$, [*Th* 50, *Cor* i]

but $AB\ AX=AX^2+AX\ XB$, [*Th* 50, *Cor* ii]

and $AB\ XB=XB^2+AX\ XB$, [*Th* 50, *Cor* ii]

$AB^2=AX^2+XB^2+2AX\ XB$

2 $AY \cdot YB = XY^2 - AX^2$. [*Th* 53, *Cor*]

Also $AY \cdot YB = 8AX^2$ [*Hyp*]

$XY^2 = 9AX^2$, whence $XY = 3AX$

$AY = 4AX = 2AB$

3. If the given lines are unequal, they can be placed as AX, XB in the Fig of Theor 52 Completing the Fig we see that sum of sqq on AX, XB exceeds the sum of the rects BG, DG by the fig AC and that this excess can never vanish since the lines are by hypothesis unequal Also each of the rects BG and DG is contained by the two given lines Hence the theorem follows

If the given lines are equal, it is easily seen that the sum of the squares is *equal* to the sum of the two rects

From the formula $(a-b)^2 = a^2 + b^2 - 2ab$ we have, by transposition, $a^2 + b^2 = 2ab + (a-b)^2$ Now when a and b are unequal, $(a-b)^2$ is necessarily positive, and $a^2 + b^2 > 2ab$

When $a = b$, $(a-b)^2 = 0$, and $a^2 + b^2 = 2ab$

Thus $a^2 + b^2$ is never less than $2ab$

4 Substituting as required, we obtain $xy = \left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2$

This can be enunciated thus

Any rectangle is equal to the excess of the square on half the sum of its length and breadth over the square on half the difference of its length and breadth.

5 (i) We have $AY \cdot YB = AX^2 - XY^2$ [*Th* 53, *Cor*]

Since AX is of constant length, $AX^2 - XY^2$ diminishes as XY increases, that is, as Y moves away from X

(ii) Let $AY = a$ units and $YB = b$ units Then $AB = a + b$

Hence $AX = \frac{a+b}{2}$, and $XY = AY - AX = \frac{a-b}{2}$

As Y moves along AB the values of a, b change, but their sum is constant Now from the given formula, remembering that $\frac{a+b}{2}$ is constant, we see that ab diminishes as $\frac{a-b}{2}$ increases that is, $AY \cdot YB$ diminishes as XY increases, or as Y moves away from X.

6 (ii) $AY^2 + YB^2 = AB^2 + 2AY \cdot YB$ [*Th* 52]

$= 4AX^2 + 2(XY + AX)(XY - AX)$

$= 4AX^2 + 2(XY^2 - AX^2)$ [*Th* 53]

$= 2AX^2 + 2XY^2$

7 Take X the middle point of AB Then, by Ex 6 (i),

$$AY^2+YB^2=2AX^2+2XY^2$$

As Y moves from A to X, the value of XY decreases from AX to zero, and AY^2+YB^2 decreases from $4AX^2$ to $2AX^2$

As Y moves from X to B, the value of XY increases from zero to XB, which is equal to AX Hence AY^2+YB^2 increases from $2AX^2$ to $4AX^2$

8 Let ABC be a △ rt angled at B, and let BD be drawn perp to AC

Then $AC^2=AD^2+DC^2+2AD\ DC$ [*Th* 51]

But $AC^2=AB^2+BC^2$ [*Th* 29]

$=AD^2+BD^2+DC^2+BD^2$ [*Th* 29]

$AD^2+DC^2+2BD^2=AD^2+DC^2+2AD\ DC$,

or $BD^2=AD\ DC$

9 Let ABC be an isosceles △, having its vertex at A, and let AY be drawn to any pt Y in BC, or BC produced

Draw AP perp to BC Then, by Theor 17, BP=PC

Thus BY, YC are the sum and difference of PY and PC

(i) When Y is an internal point.

We have $PC^2-PY^2=BY\ YC$ [*Th* 53]

$(AP^2+PC^2)-(AP^2+PY^2)=BY\ YC$,

$AC^2-AY^2=BY\ YC$ [*Th* 29]

Hence $AC^2=AY^2+BY\ YC$

(ii) When Y is in BC produced

In this case $PY^2-PC^2=BY\ YC$ [*Th* 53]

Proceeding as in (i) we obtain

$$AY^2-AC^2=BY\ YC$$

$$AC^2=AY^2-BY\ YC$$

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1 From the △APB, $AP^2+BP^2=2AO^2+2OP^2$ [*Th* 56]

Here AO=4 cm, OP=5 cm, whence $AP^2+PB^2=82$ sq cm

2 Substituting the lengths of AB, AC, BX in the relation

$AB^2+AC^2=2BX^2+2XA^2$ [*Th* 56]

we find that AX=8 5 cm, AX=XB=XC

Hence $\angle XBA=\angle XAB$, and $\angle XCA=\angle XAC$,

$\angle BAC=\angle ABC+\angle ACB$

$\angle BAC$ is a rt angle [*Th* 16]

3 Let P be the vertex and O the middle point of the base, then by Theor 56 $122=2(OP^2+25)$ [*Th* 56]

$\therefore 2OP^2=72$ whence $OP=6$ cm.

P lies on a circle of radius 6 cm whose centre is at the mid-pt of the base

4. Let ABCD be the par^m. and let the diags meet at O

Then O is the middle point of AC and BD

Then from $\triangle ABC$ $AB^2+BC^2=2AO^2+2OB^2$ [*Th* 56]

Also, from $\triangle CDA$, $CD^2+DA^2=2OC^2+2OD^2$

By addition remembering that $AO=OC$ and $OB=OD$

$$AB^2+BC^2+CD^2+DA^2=4AO^2+4OB^2$$

$$=AC^2+BD^2 \quad [Ex\ 1\ p\ 225]$$

By the above since the sum of the squares on the 4 sides $=36$ sq. in $\therefore$ the sum of the squares on the diagonals $=36$ sq in

$\therefore$ sq on longer diagonal $=27$ sq in, $\therefore$ length of that diagonal is 5 20".

5 Let ABCD be the quad and P Q, R, S the middle points of the sides AB BC, CD, DA

Then PQRS is a par^m [*Ex* 7, p 64]

and $AC=2PQ$, also $BD=2SP$ [*Ex* 3, p 64]

$AC^2+BD^2=4PQ^2+4SP^2$

$=2\{PQ^2+SR^2+SP^2+QR^2\}$

$=2\{PR^2+SQ^2\}$ [*Ex* 4]

6 (i) Let ABCD be the rect. and O the given point within it Let the diagonals meet at P Then $AC=BD$ [*Th* 4], and the diags. bisect one another $\therefore AP=BP$

Then from $\triangle AOC$ $OA^2+OC^2=2[AP^2+OP^2]$, [*Th* 56]

and from $\triangle BOD$ $OB^2+OD^2=2[BP^2+OP^2]$ $\cdot$

$\therefore OA^2+OC^2=OB^2+OD^2$

(ii) Since $OA^2+OC^2=2OP^2+2AP^2$, and $OA^2+OC^2=21\frac{1}{4}$

$\therefore 2OP^2+2AP^2=21\frac{1}{4}$ so that $2OP^2=21\frac{1}{4}-2AP^2$

Now $AC^2=AB^2+BC^2=36+6\,25=42\,25$

and $AC^2=4AP^2$ $\therefore 2AP^2=21\,125$

But $2OP^2=21\,25-2AP^2=21\,25-21\,125=0\,125$

$\therefore OP^2=0{\cdot}0625$, hence $OP=0\,25"$

7 Let ABCD be the quad', and X, Y the middle points of BD, AC Join YB, YX, YD

Then from the $\triangle$ ABC, $AB^2+BC^2=2AY^2+2BY^2$, [*Th* 56]

also from the $\triangle$ ADC, $AD^2+DC^2=2AY^2+2DY^2$

$$AB^2+BC^2+AD^2+DC^2=4AY^2+2(BY^2+DY^2)$$
$$=AC^2+4(DX^2+XY^2) \quad [\textit{Th } 56]$$
$$=AC^2+BD^2+4XY^2$$

8 We have $AB^2=AC^2+BC^2-2AC \cdot CE$ [*Th* 55]

Also $AC^2=AB^2+BC^2-2AB \cdot BF$

By addition $AB^2+AC^2=AB^2+AC^2+2BC^2-2AB \cdot BF-2AC \cdot CE$,

whence $AB \cdot BF+AC \cdot CE=BC^2$

9 Let AX, BY, CZ be the medians of the $\triangle$ ABC

Then $AB^2+AC^2=2AX^2+2BX^2$, [*Th* 56]

and $AB^2+BC^2=2BY^2+2CY^2$,

also $BC^2+AC^2=2CZ^2+2AZ^2$

By addition

$$2AB^2+2BC^2+2AC^2=2(AX^2+BY^2+CZ^2)+2BX^2+2CY^2+2AZ^2,$$
$$4AB^2+4BC^2+4AC^2=4(AX^2+BY^2+CZ^2)+4BX^2+4CY^2+4AZ^2$$
$$=4(AX^2+BY^2+CZ^2)+BC^2+AC^2+AB^2$$

Hence $3[AB^2+BC^2+AC^2]=4[AX^2+BY^2+CZ^2]$

10 Let AX, BY, CZ be the medians, intersecting at O

Then $OA=2OX$, $OB=2OY$, $OC=2OZ$ [III p 97, *Cor*],

and from the $\triangle$ BOC, $OB^2+OC^2=2BX^2+2OX^2$

Again from $\triangle$ COA, $OC^2+OA^2=2CY^2+2OY^2$,

also from $\triangle$ AOB, $OA^2+OB^2=2AZ^2+2OZ^2$

by addition

$$2OA^2+2OB^2+2OC^2=2BX^2+2CY^2+2AZ^2+2OX^2+2OY^2+2OZ^2,$$

and doubling these equals, we have

$$4OA^2+4OB^2+4OC^2=4BX^2+4CY^2+4AZ^2+4OX^2+4OY^2+4OZ^2$$
$$=BC^2+CA^2+AB^2+OA^2+OB^2+OC^2$$

Hence $3[OA^2+OB^2+OC^2]=BC^2+CA^2+AB^2$

11 Theorem 56 is true however close the vertex C is to the base AB on either side of it We conclude that it is therefore also true in the intermediate position when C actually lies at Y on AB

Thus the line AB bisected at X and divided at Y may be regarded as a $\triangle$ AYB with its base AB bisected at X

Hence $AY^2+YB^2=2AX^2+2XY^2$

12. Draw AD perp to BC, and of the $\angle^s$AXB, AXC let $\angle$AXB be that which is obtuse

$$m\mathrm{AB}^2 = m\mathrm{BX}^2 + m\mathrm{XA}^2 + 2m\mathrm{BX}\ \ \mathrm{XD}, \qquad [Th\ 54]$$

and
$$n\mathrm{AC}^2 = n\mathrm{CX}^2 + n\mathrm{XA}^2 - 2n\mathrm{CX}\ \ \mathrm{XD} \qquad [Th\ 55]$$

By addition, remembering that $m\mathrm{BX} = n\mathrm{CX}$, we have

$$m\mathrm{AB}^2 + n\mathrm{AC}^2 = m\mathrm{BX}^2 + n\mathrm{XC}^2 + (m+n)\,\mathrm{AX}^2$$

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3 Since the chords AB, CD intersect at X,

$$\mathrm{AX}\ \ \mathrm{XB} = \mathrm{CX}\ \ \mathrm{XD} \qquad [Th\ 57]$$

$$\mathrm{XD} = \frac{\mathrm{AX}\ \ \mathrm{XB}}{\mathrm{CX}} = \frac{1\ 8 \times 1\ 2}{2\ 7} = 0\ 8''$$

4 (i)
$$\mathrm{XT}^2 = \mathrm{XA}\ \ \mathrm{XB} \qquad [Th\ 58]$$
$$= 0\ 6 \times 2\ 4 = 1\ 44$$
$$\mathrm{XT} = \sqrt{1\ 44} = 1\ 2''$$

(ii) Here $\mathrm{XB} = \dfrac{\mathrm{XT}^2}{\mathrm{XA}} = \dfrac{(7\ 5)^2}{4\ 5} = 12\ 5$ cm

5. If the circle is completed and MX is produced to meet the $\bigcirc^{ce}$ again in N, then MX = XN [*Th* 31]

$$\mathrm{AX}\ \ \mathrm{XB} = \mathrm{MX}\ \ \mathrm{XN}\ [Th\ 57] = \mathrm{MX}^2$$

Hence (i) $\qquad \mathrm{XB} = \dfrac{\mathrm{MX}^2}{\mathrm{AX}} = \dfrac{4}{2\ 5} = 1\ 6''$,

whence $\qquad \mathrm{AB} = \mathrm{AX} + \mathrm{XB} = 4\ 1''$

And (ii) since $\mathrm{XB} = \mathrm{AB} - \mathrm{AX} = 7\ 4 - 4\ 9 = 2\ 5$ cm,

$$\mathrm{MX}^2 = \mathrm{AX}\ \ \mathrm{XB} = 4\ 9 \times 2\ 5 = 12\ 25 \text{ sq cm}$$

whence $\qquad \mathrm{MX} = \sqrt{12\ 25} = 3\ 5$ cm

6. (i) Let O be the centre of the $\bigcirc$, then, as in Theor 57, we have

$$\mathrm{PX}\ \ \mathrm{XQ} = r^2 - \mathrm{OX}^2,$$
$$\mathrm{OX}^2 = r^2 - \mathrm{PX}\ \ \mathrm{XQ} = 16 - 12 = 4 \text{ sq cm}$$

OX = 2 cm, and the locus of X is a concentric circle of radius 2 cm

(ii) X being outside the $\bigcirc$, we have as in Theor 58

$$\mathrm{PX}\ \ \mathrm{XQ} = \mathrm{OX}^2 - r^2$$
$$\mathrm{OX}^2 = \mathrm{PX}\ \ \mathrm{XQ} + r^2 = 20 + 16 = 36 \text{ sq cm}$$

OX = 6 cm, and the locus of X is a concentric circle of radius 6 cm

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1 The circle described on AB as diameter will pass through C [*Ex* 1, p 165] $AD \cdot DB = CD^2$ [*Ex* 5, p 235]

2 Let PXQ be the common chord

Then $AX \cdot XB = PX \cdot XQ$, from the ○ APBQ, [*Th* 57]

$= CX \cdot XD$, from the ○ CPDQ

3 Let XP, XQ be the two tangents drawn from X to the ○, and let XAB be a secant through X,

then $XP^2 = XA \cdot XB = XQ^2$ [*Th* 58]

$XP = XQ$.

4. Let AB be the common chord, X a point in AB produced, and XP, XQ the tangents from X to the two ○s

Then $XP^2 = XA \cdot XB$ [*Th* 58] $= XQ^2$

$XP = XQ$

5 Let AB produced cut PQ in X

Then $XP^2 = XA \cdot XB$ [*Th* 58] $= XQ^2$

$XP = XQ$.

6 If the circle through A, B, C does not also pass through D, suppose it to cut CD again at E

Then $CX \cdot XD = AX \cdot XB$ [*Hyp*]

$= CX \cdot XE$ [*Th* 57]

$XD = XE$

Hence E coincides with D, and the points A, B, C, D are concyclic.

7 Because the ∠s AQB, APB are rt angles,

A, B, Q, P are concyclic [*Th* 39 Converse]

$AO \cdot OP = BO \cdot OQ$ [*Th* 57]

8. Because the ∠CDB is a rt angle, the ○ on CB as diameter passes through D, and ∠ACB being a rt angle, the side AC must touch this ○ at C

$AD \cdot AB = AC^2$ [*Th* 58]

9 Let CA be the diameter of the first ○ cutting the second in E, and DA the diameter of the second cutting the first in F

Then the ∠s CFA, DEA are rt ∠s, [*Th* 41]

C, F, E, D are concyclic.

$CA \cdot AE = DA \cdot AF$ [*Th* 57]

10 Let PK, PT be the two tangents. Then by Ex 7, p 177 the $\angle$PQK is a rt. $\angle$. $\therefore$ the $\bigcirc$ on PK as diameter passes through Q. Also, since $\angle$OKP is a rt $\angle$, the radius OK must touch this $\bigcirc$ at K. $\therefore$ OP . OQ $=$ OK^2 [*Th* 58] $= r^2$

11 Let CD cut AB at X. Join BQ.

Then since the $\angle$s PXB, PQB are rt $\angle$s, the pts P, Q, B, X are concyclic.

$$AP \cdot AQ = AX \cdot AB \qquad [Th\ 58]$$
$$= \text{a constant}$$

12 Draw AX perp to CD to meet it at X. Let R be a pt in AX such that AX . AR $=$ the given constant. Then R is a fixed point.

But, since AP . AQ $=$ AX . AR, it can be shewn, as in Ex 6, that P, X, R, Q are concyclic. Hence $\angle$AQR $= \angle$PXR [*Ex* 5, p 163] $=$ a rt $\angle$, from which it follows that the locus of Q is the $\bigcirc$ on AR as diameter.

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1 Let PQ be the given chord, R its mid-pt, and RN the perp at R to PQ, meeting the arc in N and passing through the centre O. Then if NR be produced to meet the $\bigcirc^{ce}$ again in M, we have NM $= 2r$, and RM $= 2r - h$.

$$\text{Also} \quad NR \cdot RM = PR \cdot RQ; \qquad [Th\ 57]$$
$$\text{that is,} \quad h\,(2r-h) = c^2$$

Substituting $c = 12$, and $h = 8$, we obtain $r = 13$.

2. (i) Put $h = 18$, $r = 25$ in the formula of Ex 1.

Then $c^2 = 18 \times 32$; whence $c = 24$.

Hence the span, measured by $2c$, is 48 feet.

(ii) Putting $h = 10$, $r = 25$ in the formula, we obtain $c = 20$.

Thus the new span is 40 ft, and the reduction is 8 ft.

3 Substituting the given values we obtain $h(34 - h) = 64$.

Hence $h^2 - 34h + 64 = 0$, whence $h = 2$ or 32.

Thus the height of the arc is either 2 cm. or 32 cm, the first result referring to the minor and the second result to the major of the two arcs cut off by the chord.

4 Let X be the external pt, O the centre of the given ○, XAB the diam through X, and XT the tangent from X to the ○

Then by Th 37, XA is the shortest distance from X to the ○

$$XA=d, \quad XB=d+2r, \quad XT=t$$

But $\quad XA \cdot XB=XT^2,$ [*Th* 58]

that is, $\quad d(d+2r)=t^2$

By substituting $d=1{\cdot}2$, $t=2{\cdot}4$ in the above formula, we obtain $r=1{\cdot}8$ Thus the diameter $=3{\cdot}6'$

5. (i) Consider a section of the earth through the top of the cliff, P, and the centre of the earth O The section will be circular, and if PT be the tangent from P, then T will be a point on the horizon Let PO cut the circle in A and B

Now $PT=22\frac{1}{2}$ miles, and $PA=330$ ft $=\frac{1}{16}$ mi

Also, by Theor 58, $PT^2=PA \cdot PB$,

$$PB=\frac{PT^2}{PA}=\frac{45}{2}\times\frac{45}{2}\times 16=8100 \text{ mi}$$

$$AB=8100 \text{ mi} -330 \text{ ft}$$

$$=8100 \text{ mi}, \text{ approximately}$$

(ii) In the Fig of (i) we have now $PA=66$ ft $=\frac{1}{80}$ mi, $AB=8100$ mi

$PT^2=PA \cdot PB$ [*Th* 58]

$$=\tfrac{1}{80}(8100+\tfrac{1}{80}) \text{ sq mi}$$

$$=101{\cdot}25 \quad \text{sq mi (approx)}$$

whence $PT=10$ *mi approx*

6. Let APB be the given arc and P its mid-pt From P draw PN perp to AB then, by Theors 45, 18, PN bisects AB, and will therefore when produced pass through the centre of the ○ Let PN meet the ○^{ce} again at D

Then $\quad b^2=AP^2=PN^2+AN^2$ [*Th* 29]

$=PN^2+PN \cdot ND$ [*Th* 57]

$=(PN+ND)PN=PD \cdot PN=2rh$

7 From P draw PK perp to AB and meeting it at K

Then because the ∠s PKB, PCB are rt ∠s, P, K, B, C are concyclic [*Th* 40 *Converse*]

$\therefore \quad AK \cdot AB=AP \cdot AC$ [*Th* 58]

Similarly $BK.BA = BP.BD$

Now $AB^2 = (AK+KB)AB$

$$= AK.AB + BK.BA$$

$$= AP.AC + BP.BD$$

8 Let AE, DF meet at P on the line of centres [*Ex* 8, p 187], and let the line of centres meet BC at O. Then O is the mid-pt of BC (III, p 143)

By Ex 7, p 187, AE = DF, and by Ex 5, p 236, G and H are the mid-pts of AE, DF; thus EG = FH and ∴ PG = PH [*Th* 17, *Cor*]

Hence $\triangle$ PGH is isosceles, and ∴ PO which is perp to the base GH bisects it [*Th* 18],

∴ O is the mid-pt of GH

$$\therefore GH^2 - BC^2 = 4GO^2 - 4BO^2 \qquad [Ex\ 1,\ p\ 225]$$

$$= 4(GO^2 - OB^2) = 4(GO+OB)(GO-OB)$$

$$= 4GC.GB \qquad [Th\ 53]$$

$$= 4GA^2$$

$$= AE^2$$

$$\therefore GH^2 = BC^2 + AE^2$$

9 Let O be the centre of the ○, and PT the tangent from P

Then $PM^2 = OP^2 - OM^2 = PT^2 + OT^2 - OM^2$ [*Th* 29]

$$= PC.PD + OA^2 - OM^2 \qquad [Th\ 58]$$

$$= PC.PD + AM.MB \qquad [Th\ 53]$$

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1, 2 Apply Prob 32

3 A convenient rectangle is one whose length = 2.5″ and whose breadth = 1.5″. Each side of the square must be $\sqrt{3.75}$ in., or 1.94″

4 Let ABC be the equilateral $\triangle$, AD the perp from A on BC. Through A, C draw AK, CK par[1] to BC, AD respectively. Then ADCK is the req[d] rect. Now proceed as in Prob 32

5 Apply Probs 17 and 32

6 See Fig and Construction of I, p 244 Take AB$=9$ cm and DE$=4$ cm

Here AX$+$XB$=$AB$=9$, and AX XB$=$DE$^2=16$,

if then we denote the lengths of AX, XB by x and y, we have the two equations $x+y=9$, $xy=16$

Now, by measurement, AX$=6\ 6$ cm, XB$=2\ 4$ cm

Hence $x=6\ 6$, $y=2\ 4$ is a solution of the given equations

7 In the Fig on p 244 take AB$=4\ 0'$ and DE$=1\ 3''$

Then by measurement, AX$=3\ 52''$ and XB$=0\ 48''$ But $0\ 1''$ represents 1 unit

if AX$=x$, XB$=y$ we have $x=35\ 2$, $y=4\ 8$

8 Draw AB$=7\ 2$ cm, at B set up BF perp to AB and$=5\ 0$ cm, join AF, and draw FE at rt $\angle^s$ to AF meeting AB produced in E Then by Theor 41 the semicircle on AE passes through F, as in Prob 32

rect AB BE$=$BF$^2=25$ sq cm

By measurement BE$=3\ 5$ cm

9 See Fig and Constr of II, p 245 Take AB$=8$ cm, BF$=6$ cm

Now AX$-$XB$=$AB$=8$ cm

and AX XB$=$BF$^2=36$ sq cm

the no of cm in AX, XB, viz 11 2 and 3 2, give the solution of the equations $x-y=8$, $xy=36$

10 Draw a line AB 10 cm in length On AB describe a semicircle At B set up BK perp to AB and 2 5 cm in length Through K draw KP par^l to AB to meet the O^{ce} at P Join AP, PB From P draw PX perp to AB

Then AP$^2+$PB$^2=$AB^2 [*Th* 41 and 29]$=100$,

and AP PB$=2\ \triangle$ APB (since $\angle$APB$=90°$)$=$PX AB$=25$

the values of x and y are given by the numbers of cm in AP, PB, which are found by measurement to be 9 6 and 2 6 respectively

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1 Use the Construction of Prob 33

Denote the length of the greater segment by x inches, and that of the less by $(4-x)$ inches

$$4(4-x)=x^2, \text{ or } x^2+4x=16$$

From this we get $x=-2\pm2\sqrt{5}$, and for *internal* division x must be positive, hence the req^d value is $2\sqrt{5}-2$, or 2 47

2. For Construction, see the Note and Fig of p 241

Let the segment measured from A contain x inches, and let that measured from B contain $2-x$ inches

Then for medial section, $2(2-x)=x^2$

From this quadratic, $x=-1\pm\sqrt{5}$, or 1 24 and -3 24

For *external* division the segment AX′ is taken in the sense opposite to AB, this fact is represented by the negative sign Hence AX′$=-3$ 24″ nearly

3 By Theor 29, $AC^2=AB^2+BC^2=a^2+\frac{a^2}{4}=\frac{5a^2}{4}$

$$AC=\frac{a\sqrt{5}}{2}$$

Hence (i) $AX=AD=AC-CB=\frac{a\sqrt{5}}{2}-\frac{a}{2}$, (Fig p 240)

(ii) $AX'=AD=AC+CB=\frac{a\sqrt{5}}{2}+\frac{a}{2}$ (Fig p 241)

The fact that AX′ is to be measured in the opposite direction to AB is expressed by affixing a negative sign (p 132) Thus the direction and magnitude of AX′ are both given by the statement $AX'=-\left(\frac{a\sqrt{5}}{2}+\frac{a}{2}\right)$

4 Let AB be divided in medial section at X

From AX cut off $AX'=BX$

Then we have $AB \cdot BX=AX^2$, and $AX'=BX$

Now $AB \cdot BX=(AX+XB)XB=(AX+AX')AX'=AX \cdot AX'+AX'^2$,

And $AX^2=(AX'+X'X)AX=AX \cdot AX'+AX \cdot XX'$,

$AX \cdot AX'+AX'^2=AX \cdot AX'+AX \cdot XX'$,

whence $AX'^2=AX \cdot XX'$

That is, AX is divided in medial section at X′

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1 Let A be the vertical angle Then each of the base angles is 2A $A+2A+2A=180°$ [*Th* 16], whence $A=36°$

2 Let BAD be a rt angle On AB by Prob 34, construct an isosceles △ BAC having each of the ∠s at B, C double the ∠A

Bisect ∠ BAC by AK

Then ∠ BAC$=36°$, by Ex 1

∠ BAK$=18°$=one-fifth of a rt ∠

3 The $\triangle$ AXC is such a $\triangle$

As in Prob 34, $\angle ABC = 2\angle BAC$, and $\angle BCX = \angle BAC$

Now $\angle AXC = \angle ABC + \angle BCX$, [*Th* 16]

$= 3\angle BAC$

Also $\angle^s$ XAC, XCA are equal [*Prob* 34]

$\triangle$ AXC is isosceles and its vertical angle is three times either angle at the base

To construct the $\triangle$, *take any centre* A, *and with any radius* AB *describe a* $\bigcirc$ Divide AB at X so that

$AB \cdot BX = AX^2$ [*Prob* 33]

With centre X, and radius XA, cut the first $\bigcirc$ in C

Then AXC is the req^d $\triangle$

4 Let $AB = a$ Then $BC = XA = a\,\dfrac{(\sqrt{5}-1)}{2}$. [*Prob* 33 *Note*]

$$\frac{BC}{AB} = \frac{\sqrt{5}-1}{2}$$

5 (i) Because BC touches the $\bigcirc$ ACF, $\angle ACB = \angle AFC$

But $\angle ABC = \angle ACB$, and $\angle ACF = \angle AFC$, [*Th* 5]

whence, by Theor 16, the $\angle BAC = \angle CAF$

$BC = CF$ [*Th* 42 and 45]

(ii) The $\triangle^s$ BAC, XAC have equal bases BC, XC and equal vertical angles BAC, XAC

their circum-circles are equal [*Ex* 2, p 206]

(iii) By (i) the $\angle^s$ BAC, CAF are equal, and by Ex 1 each must be 36°, or one-tenth of 360°

Hence, by Prob 30, BC, CF are sides of a regular decagon inscribed in the $\bigcirc$ BCD

(iv) By (i) and Ex 1, each of the $\angle^s$ ACX, XAC, CAF is 36°

each of the arcs AX, XC, CF subtends 36° at the $\bigcirc^{ce}$ of $\bigcirc$ AXC

each of the arcs AX, XC, CF subtends 72° at the *centre* of $\bigcirc$ AXC

But 72° is one-fifth of 360°

, as in Prob 30, AX, XC, CF are sides of a regular pentagon inscribed in the $\bigcirc$ AXC

6. Take S′ the mid-pt of the arc XC. Join AS′, S′C.
Then because the arc XS′ = arc S′C, AS′ bisects the $\angle BAC$.
But $\angle XAS' = \angle XCS'$ [*Th.* 39], and $\angle BAC = \angle XCB$ [*Prob.* 34],
CS′ bisects the $\angle XCB$.
Now since the $\triangle^s$ BAC, BCX are both isosceles, the bisectors of their vertical angles will also bisect their bases at rt. $\angle^s$. [*Th.* 4.]
AS′ and CS′ will, if produced, bisect BC and BX at rt. $\angle^s$.
S′ is the circum-centre of the $\triangle CBX$.

7. By the last example S′ is the intersection of the bisectors of the $\angle^s$ BAC, BCX.
Also I is the intersection of AS′ and CX, for these bisect the $\angle^s$ BAC, ACB respectively.
Again I′ is the intersection of BI and CS′, the bisectors of the $\angle^s$ XBC, XCB.
From the $\triangle IAB$, the ext. $\angle S'II' = \angle IAB + \angle IBA$
$= \frac{3}{2}$ of $\angle BAC$. [*Prob.* 34.]
Also from the $\triangle I'BC$, the ext. $\angle S'I'I = \angle I'BC + \angle I'CB$
$= \angle BAC + \frac{1}{2} \angle BAC$
$= \frac{3}{2}$ of $\angle BAC$.
$\angle S'II' = \angle S'I'I$, that is, $S'I = S'I'$.

8. Let AB be divided in medial section at X, so that $AB \cdot BX = AX^2$.
Then $(AX+XB)(AX-XB) = AX^2 - XB^2$ [*Th.* 53.]
$= AB \cdot BX - XB^2$ [*Hyp.*]
$= (AB - BX)BX$
$= AX \cdot XB$.

9. Since $(AB - BX)^2 = AX^2$,
$AB^2 + BX^2 = AX^2 + 2AB \cdot BX$ [*Th.* 52.]
$= AX^2 + 2AX^2 = 3AX^2$. [*Hyp.*]

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1. Since the sum of the roots is 10 and their product is 16, the solution will be given by dividing a line of 10 cm. into two segments such that the rectangle contained by them is 16 sq. cm. Apply I. p. 244.
Examples 2, 3 are done similarly.

4 Divide a line 5 cm long externally, so that the rectangle contained by the segments is 36 sq cm Apply II p 245

5 The solution is similar to that of Ex 4

6 Proceed as in I p 244, taking $AB=10$ cm, and $DE=\sqrt{20}$ cm See Prob 32

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1 Let A be the pt (0, 4) and B the pt (0 9)

Then, by Theor 58, $OP^2=OA \cdot OB=4\times 9=36$

$$OP=6$$

2 Let A be the pt (9, 6) From A draw AB perp to OY and meeting it at B Then OB is the tangent from O to the circle Hence rect contained by the segments of any chord through $O=OB^2=36$

Again let P be the pt (9, 12), and CPAD the diameter through P

Then the rect $CP \cdot PD$ is required [*Th* 57]

Now $CP=CA-AP=3$, and $PD=PA+AD=15$

$$CP \cdot PD=45$$

3 Let A, B, C be the pts in the given order, and let OY cut the ○ again at D

$$OC \cdot OD=OA \cdot OB \qquad [Th\ 58]$$

$$OD=\frac{6\times 24}{9}=16$$

Also, if OT be the tangent from O to the ○,

$$OT^2=OA \cdot OB=144, \text{ whence } OT=12$$

4. Let A, B, C be the pts in the given order A circle can be drawn through A, B, C by Prob 25

Also $OA^2=100=5\times 20=OB \cdot OC$, and OA touches the ○ at A. [*Th* 59]

Let P be the centre of the ○, and from P draw PK perp to BC

K is the mid-pt of BC [*Th* 31] Hence $BK=7.5$

$$PA=OK=OB+BK=12\tfrac{1}{2}$$

coords of P are $(10, 12\tfrac{1}{2})$, and radius of $○=PA=12\tfrac{1}{2}$

5. Let A, B, C [illegible] CY [illegible] D.

∴ CO . CD = CA . CB [illegible]; hence $CD = \frac{CA . CB}{CO} =$ [illegible]

∴ [illegible]

[illegible] AB [illegible]
[illegible]

[illegible] CO [illegible]
[illegible]

∴ [illegible]

[illegible] = CA . CB = [illegible]

∴ [illegible] = [illegible]

6. [illegible] CA . CB = [illegible] CO . CD = [illegible],

∴ CA . CB = CO . CD;

∴ [illegible] A, B, C, D [illegible] [illegible]

[illegible] AO, BO.

Then $CA^2 - CO^2 - CB^2 - CD^2 = AO^2 - BO^2$ [illegible]

[illegible] AB, CD [illegible]

[illegible] AO, BO [illegible] [illegible]

∴ $AO^2 - BO^2 =$ [illegible] [illegible]

∴ $CA^2 + CB^2 - CO^2 - CD^2 =$ [illegible]

[illegible]

7. Let A and B [illegible]

[illegible] AB [illegible] C [illegible]
[illegible]

[illegible] AB [illegible] AB [illegible] G. Then
G [illegible]

Let O [illegible] OX.

Then OB . OD = OA^2 [illegible] whence OD = [illegible] = [illegible]

∴ D [illegible]

From C draw CM [illegible] BD, [illegible]
[illegible] M.

Then [illegible] = CA − CM = (CB − CD) = [illegible]

8 Let A be the point (0, 8), M and M′ the pts (0, 13), (0, −13) respectively

Then the centre of the reqd ○ must lie on one of the two lines drawn through M, M′ parl to OX

It must also lie on a ○, centre A, radius = 13

The intersection of these two loci give two positions of the centre, P and Q, each lying on the parallel through M

Since the common chord of two ○s is perp to the line of centres, the second pt of intersection, B, of the two circles also lies on OY

Let the ○ whose centre is P touch OX at C

$$OA \cdot OB = OC^2 = PM^2 = AP^2 - AM^2 = 13^2 - 5^2 = 144$$

$$OB = \frac{144}{8} = 18\text{, whence } AB = OB - OA = 10$$

9 Draw two st lines each parallel to OX and at a distance 15 from it on either side With centre O, and radius 15 + 15 *i e* 30, describe a ○ This cuts each of the two lines in two pts Any one of these *four* pts is a possible position of the centre of the required ○ [Proof by Loci (iv) and (v), p 188] Hence there are four possible circles

Let A be that centre which lies in the 1st quadrant, and AM its perp on OX. Then OA = 30 and AM = 15

$OM^2 = OA^2 - AM^2$, whence $OM = 15\sqrt{3} = 25\ 98$ or 26, nearly

Thus the coords of A are (26, 15)

10 Draw any circle through A and B Take any convenient point Q on this ○, and draw a ○ through C, D, Q [*Prob* 25] Let R be the 2nd pt of intersection of the two ○s, and let QR produced meet the x-axis in P

Then $PA \cdot PB = PQ \cdot PR$, [*Th* 58]

$= PC \cdot PD$

Let $OP = x$ $PA = x - a$, $PB = x - b$, $PC = c - x$, $PD = d - x$

$(x - a)(x - b) = (c - x)(d - x)$,

whence $x = (ab - cd)/(a + b - c - d)$

The numerical example is done similarly

PART V

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3 Let AB be the given line, and X the pt of *internal* division

Since AX XB=5 7, if AX is divided into 5 equal parts, then XB must contain 7 such equal parts

AB contains 5+7, or 12, such equal parts

Hence $\quad AX=\frac{5}{12}$ of $AB=\frac{5}{12}$ of $9\ 6'=4\ 0'$,

and $\quad XB=\frac{7}{12}$ of $AB=\frac{7}{12}$ of $9\ 6''=5\ 6''$

4. Let AB be the given line, and Y the pt of *external* division

Since AY YB=11 8, if AY is divided into 11 equal parts, then YB must contain 8 such equal parts

∴ AB contains 11 − 8, or 3, such equal parts

Hence $\quad AY=\frac{11}{3}$ of $AB=\frac{11}{3}$ of 4 5 cm =16 5 cm ,

and $\quad YB=\frac{8}{3}$ of $AB=\frac{8}{3}$ of 4 5 cm =12 0 cm

5 As in Exx 3 and 4 ,

For *internal* division, $AX=\frac{5}{5+3}$ of $AB=\frac{5}{8}$ of 6 4 cm = 4 0 cm

For *external* division, $AY=\frac{5}{5-3}$ of $AB=\frac{5}{2}$ of 6 4 cm =16 0 cm

These values will be found to satisfy the given formula

6 Let AB be the given line, and X the pt of *internal* division

Since $AX : XB = m : n$, if AX is divided into m equal parts, then XB must contain n such equal parts

AB contains $m+n$ such equal parts

Hence $AX = \dfrac{m}{m+n}$ of $AB = \dfrac{m}{m+n}\, a$ inches

$XB = \dfrac{n}{m+n}$ of $AB = \dfrac{n}{m+n}\, a$ inches

7 Let AB be the given line, and Y the pt of *external* division

Since $AY : YB = m : n$, if AY is divided into m equal parts, then YB must contain n such equal parts

AB contains $m-n$ such equal parts

Hence $AY = \dfrac{m}{m-n}$ of $AB = \dfrac{m}{m-n}\, a$ inches

$YB = \dfrac{n}{m-n}$ of $AB = \dfrac{n}{m-n}\, a$ inches

8 By hypothesis, $\dfrac{a}{b} = \dfrac{x}{y}$, and $\dfrac{b}{c} = \dfrac{y}{z}$

by multiplication, $\dfrac{a}{b}\,\dfrac{b}{c} = \dfrac{x}{y}\,\dfrac{y}{z}$, whence $a : c = x : z$

9 Since $\dfrac{a}{b} = \dfrac{x}{y}$, $\quad \dfrac{b}{a} = \dfrac{y}{x}$,

hence $\dfrac{b}{a} + 1 = \dfrac{y}{x} + 1$, or $\dfrac{b+a}{a} = \dfrac{y+x}{x}$

$$a+b : a = x+y : x$$

10 By hypothesis, $\dfrac{a}{b} = \dfrac{b}{c}$

Now $\dfrac{a}{c} = \dfrac{a}{b}\,\dfrac{b}{c} = \dfrac{a}{b}\,\dfrac{a}{b}$ [*Hyp*]

$$= \frac{a^2}{b^2}$$

11. (i) $\quad AX : XB = CY : YD, \quad$ or, $\dfrac{AX}{XB} = \dfrac{CY}{YD},$ [*Hyp.*]

$$\frac{AX}{XB} + 1 = \frac{CY}{YD} + 1, \text{ or, } \frac{AX+XB}{XB} = \frac{CY+YD}{YD},$$

$$\text{that is, } \frac{AB}{XB} = \frac{CD}{YD}$$

(ii) $\quad AX : XB = CY : YD$ [*Hyp.*]

$$\frac{XB}{AX} = \frac{YD}{CY} \text{ (inversely)}$$

$$\therefore \quad 1 + \frac{XB}{AX} = 1 + \frac{YD}{CY}$$

$$\text{or} \quad \frac{AB}{AX} = \frac{CD}{CY}$$

12. By hypothesis, $\quad ad = bc$

$$\text{Now} \quad \frac{a}{b} = \frac{ad}{bd} = \frac{bc}{bd} = \frac{c}{d},$$

$$a : b = c : d$$

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2 (i) We have $\quad AY : YC = AX : XB;$ [*Th.* 60]

$AY : AC = AX : AB,$ [*Th.* 60, *Cor.*]

$AY : 2.4 = 2.1 : 3.6,$

or, $\dfrac{AY}{2.4} = \dfrac{2.1}{3.6}$

$$AY = \frac{2.1}{3.6} \text{ of } 2.4 = 1.4''$$

(ii) We have $\quad BX : BA = CY : CA$ [*Th.* 60, *Cor.*]

$BX : 2.0 = 0.6 : 1.5,$

$BX = \dfrac{0.6}{1.5} \text{ of } 2.0'' = 0.8''$

(iii) $\quad AY : YC = AX : XB = 8 : 3$

$AY = \dfrac{8}{8+3} \text{ of } AC = \dfrac{8}{11} \text{ of } 8.8 \text{ cm} = 6.4 \text{ cm}$ [*Ex.* 6, p. 253]

and $YC = \dfrac{3}{8+3} \text{ of } AC = \dfrac{3}{11} \text{ of } 8.8 \text{ cm} = 2.4 \text{ cm}$

3. (i) By Theor 60, Cor, $\frac{AY}{AC}=\frac{AX}{AB}=\frac{7\ 2}{4\ 5}$,

$$AY=\frac{72}{45} \text{ of } AC=\frac{8}{5} \text{ of } 3\ 5 \text{ cm } =5\ 6 \text{ cm}$$

(ii) By Theor 60, Y divides AC externally in ratio 11 4,

$$AY=\frac{11}{11-4} \text{ of } AC=\frac{11}{7} \text{ of } 4\ 9 \text{ cm } =7\ 7 \text{ cm}$$ [*Ex* 7, p 253]

and $$YC=\frac{4}{11-4} \text{ of } AC=\frac{4}{7} \text{ of } 4\ 9 \text{ cm } =2\ 8 \text{ cm}$$

4 Let the three par^l lines cut the transversals in A, B, C and D, E, F respectively Let AB BC$=m$ n, then if AB be divided into m equal parts, BC may be divided into n such equal parts

Through the pts of division in AB, BC let par^{ls} be drawn to AD, then these par^{ls} divide DE, EF into parts which are all equal [*Th* 22]

Also of these equal parts DE contains m and EF contains n, hence DE EF$=m$ n=AB BC

5 Let P, Q be the mid-pts of the oblique sides AD, BC of the trap^m ABCD

Through P draw PX par^l to AB to meet BC in X

Then by the previous Ex, BX=XC, *i e* X coincides with Q.

PQ is par^l to AB

6 Because EF, EG are par^l to bases BA, BD,

CF FA=CE EB, [*Th.* 60]

and CG GD=CE EB

CF FA=CG GD

, from the $\triangle$CAD, FG is par^l to AD

7 Draw CK par^l to DF cutting AB in K

Then BD DC=BF FK [*Th* 60]

But $\angle$AFE=$\angle$AEF, AF=AE

And AF FK=AE EC, [*Th* 60]

FK=EC

Hence BD DC=BF CE

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2. By Theor 61, X and Y divide the base internally and externally in the ratio of BA : AC, *i e* 7·2 : 5·4, or 4 : 3

∴ $BX = \frac{4}{4+3} \cdot BC = \frac{4}{7}$ of 3·5 cm = 2·0 cm ; [*Ex* 6, p 253]

and $BY = \frac{4}{4-3} \cdot BC = 4$ of 3·5 cm = 14·0 cm [*Ex* 7, p 253]

3 (i) Let BC be the given st line On BC describe a triangle ABC, having BA double of CA [*Prob* 8].

Bisect the ∠BAC by AX meeting BC at X

Then BX : XC = BA : AC = 2 : 1

∴ BX = 2XC

Bisect BX in Y. Then BY = YX = XC

(ii) Let BC be the given st line. On BC describe a △ABC having BA, CA equal to any convenient lengths which are in the ratio 3 : 2

Bisect the ∠BAC internally and externally by AX, AY, meeting BC in X and Y Then, by Theor. 61, X and Y are the required points.

4. Because DE bisects ∠ADB,

BE : EA = BD : DA. [*Th* 61]

Because DF bisects ∠ADC,

CF : FA = CD : DA.

But BD = CD ∴ BE : EA = CF : FA.

∴ EF is par^l to BC. [*Th* 60]

5 Let the bisectors of ∠s A and C meet at X in BD

Then DA : BA = DX : XB = DC : BC [*Th* 61]

∴, *alternately*, DA : DC = BA : BC

Let Y divide AC in this last ratio Then DY, BY are the bisectors of ∠s D and B, which therefore meet in AC

6. (i) In the $\triangle$ABC let BI, CI, the bisectors of B and C, intersect in I Join AI And produce AI to cut BC in D

Then from $\triangle$ABD, BA BD=AI ID [*Th* 61]

Also from $\triangle$ACD, CA CD=AI ID

BA BD=CA CD,

or, BA CA=BD DC

AD bisects the $\angle$A [*Th* 61, *Conv*]

(ii) Let BI_1, CI_1, the bisectors of the ext $\angle^s$ B and C, intersect in I_1 Join AI_1 cutting BC in D

Then from $\triangle$ABD, AB BD=AI_1 I_1D [*Th.* 61]

Also from $\triangle$ACD, AC CD=AI_1 I_1D [*Th* 61]

AB BD=AC CD, or, AB AC=BD DC

AD bisects int $\angle$A [*Th* 61]

7 By Theor 61 we have AB AC=BX XC

, componendo, AB+AC AC=BX+XC XC=BC XC,

, alternately, AB+AC BC=AC CX

But since CI bisects the $\angle$ACX, we have AC CX=AI IX,

AI IX=AB+AC BC

8 See Ex 4, p 325

9 Let BC be the given base Divide BC internally and externally at P and Q in the ratio of the remaining sides [*Prob* 37] On PQ describe a semicircle, and on BC describe a segment containing an $\angle$ equal to the given vertical angle [*Prob* 24] The intersection of the segment and the semicircle is the req[d] vertex (See Ex 8)

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1 By Theor 14, the $\triangle^s$AXY, ABC are equiangular then corresponding sides are proportional [*Th* 62]

AY AC=AX AB, or AY 2 0=1 5 2 5,

that is, $\frac{AY}{2\,0}=\frac{1\,5}{2\,5}$, AY=$\frac{1\,5}{2\,5}$ of 2 0″=1 2″

(ii) and (iii) are done similarly

2 (i) By Theor 14, the Δ^s AXY, ABC are equiangular, and have their corresponding sides proportional [*Th* 62]

$XY \quad BC = AX \quad AB,$

or, $XY \quad 3\ 6 = 1\ 4 \quad 2\ 4,$

whence $XY = \frac{1\ 4}{2\ 4}$ of $3\ 6'' = 2\ 1''$

(ii) is done similarly.

3 Here $BR \quad BC = QR \quad AC$, whence $BR = 2\ 5''$; [*Th* 62]

and $BQ \,.\, BA = QR \quad AC$, whence $BQ = 3\ 5''$

4. By Theor 60, $AQ \,.\, AC = AP \quad AB$, whence $AQ = 2\ 8$ cm ,

and $QC = AC - AQ = 4\ 2$ cm

By Theor. 62, $PQ \quad BC = AP \quad AB$, whence $PQ = 3\ 2$ cm

6 Here $AB = AX + XB = 12\frac{3}{4}$ ft

And $\frac{BC}{XY} = \frac{AC}{AY} = \frac{AB}{AX} = \frac{12\frac{3}{4}}{8\frac{1}{2}} = \frac{3}{2}$.

$BC = \frac{3}{2}$ of $XY = \frac{3}{2}$ of $3\frac{1}{3}$ ft $= 5$ ft ,

and $AC = \frac{3}{2}$ of $AY = \frac{3}{2}$ of $6\frac{1}{6}$ ft $= 9\frac{1}{4}$ ft

7 By Theor 29, $AB^2 = 3^2 + (1\frac{1}{4})^2 = 10\frac{9}{16}$, whence $AB = 3\frac{1}{4}''$

Now $BQ \quad BC = BP \quad BA = PQ \quad AC,$

or $BQ \quad 3 = BP \quad 3\frac{1}{4} = \frac{1}{2} \quad 1\frac{1}{4} = 2 \quad 5$

$BQ = \frac{2}{5}$ of $3'' = 1\ 2''$, and $BP = \frac{2}{5}$ of $3\frac{1}{4}'' = 1\ 3''$

Also $AP = AB - BP = 1\ 95''$

8 By Theor 62, $PQ \quad BC = AP \quad AB$

By Theor 60, $AP \quad AB = AX \quad AD$

$PQ \quad BC = AX \quad AD = 5 \quad 8$

$PQ = (\frac{5}{8} \times 9)$ cm $= 5\frac{5}{8}$ cm

9 Since $CP \quad PE = BP \quad PD,$

ED is par^l to BC [*Th* 60]

Δ^s EPD, CPB are equiangular to one another [*Th* 14]

$ED \quad BC = EP \quad CP = 2 \quad 5$ [*Th* 62], whence $ED = 0\ 8$ cm

Again the Δ^s AED, ABC are equiangular to one another,

$AD \quad AC = ED \quad BC$, whence $AD = 1\ 4$ cm

And $DC = AC - AD = 2\ 1$ cm

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1. In the $\triangle$ ABC let P, Q be the mid-pts of the sides AB, AC
Then AP PB=AQ QC, PQ is par[l] to BC [*Th* 60]
the $\triangle$s APQ, ABC are equiangular, hence
PQ BC=AP AB=1 2
PQ is half of BC

2 The $\triangle$s AOB, COD are equiangular to one another,
OA OC=OB OD=AB CD=2 1
OA=2OC, and OB=2OD

3 Let PA, QB, RC meet in O Then $\triangle$s OBA, OQP are equiangular to one another
AB PQ=OB OQ. [*Th* 62]
Similarly, BC QR=OB OQ,
AB PQ=BC QR

4. The $\triangle$s FDC, DEA are each equiangular to $\triangle$ FEB [*Th* 14]
From the $\triangle$s DEA, FEB, DA FB=AE BE [*Th* 61]
alternately, DA AE=FB BE
Similarly FC CD=FB BE
DA AE=FB BE=FC CD

5 Join BD Then DE=EA, and BG=GA,
GE is par[l] to BD [*Th* 60], and $\triangle$s AGE, ABD are equiangular to one another
AG AB=GE BD,
but AG is half of AB, GE is half of BD
Similarly HF is half of BD, GE=HF

6 Because $\triangle$s ABF, CEF are equiangular to one another,
EF BF=EC BA
And because $\triangle$s ABG, EDG are equiangular,
EG AG=ED AB
But CE=ED
EF BF=EG AG
GF is par[l] to AB [*Th* 60]

7. In the △ˢ CAB, BAD, ∠ACB = rt ∠ [*Th.* 41] = ∠ABD [*Th* 46],
the ∠A is common, third ∠ˢ ABC ADB are equal [*Th* 16]
the △ˢ are equiangular to one another
. AC AB = AB AD [*Th* 62]
rect AC, AD = AB^2 [*Th* III p 250]
= a constant

8 In the △ˢ AXC, DXB,
the ∠A = the ∠D, in the same segment,
the ∠C = the ∠B, in the same segment,
and ∠AXC = ∠DXB; [*Th* 3]
. the △ˢ are equiangular to one another,
AX DX = XC XB, [*Th.* 62]
rect AX XB = rect DX XC [III p 250]

9 In the △ˢ AXT, TXB,
∠XTA = ∠TBX, in the alt segment,
and the ∠ at X is common to both,
the third ∠ˢ are equal [*Th* 16], and the △ˢ are equiangular .
XA XT = XT XB; [*Th* 62]
. rect XA XB = XT^2 [III p 250]

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1 Let PQ any line par[l] to BC, cut AB, AC in P, Q and the median AD in X
Then △ˢ AXP, ADB are equiangular to one another,
PX BD = AX AD
Also △ˢ AXQ, ADC are equiangular to one another,
XQ DC = AX AD,
PX BD = XQ DC, and BD = DC, PX = XQ.

2 (i) Let AD, A′D′ be the perps from A, A′ on BC, B′C′.
Then ∠ABD = ∠A′B′D′ [*Hyp*], and ∠ADB = ∠A′D′B′, being rt ∠ˢ
△ˢ ABD, A′B′D′ are equiangular to one another, [*Th* 16]
. AD A′D′ = AB A′B′ = ratio of a pair of corresponding sides

(ii) Let O, O′ be the two circum-centres

Then $\angle BOC = 2\angle BAC$ [*Th* 38]

$= 2\angle B'A'C'$ [*Hyp*]

$= \angle B'O'C'$ [*Th* 38]

And the $\triangle^s$ BOC, B′O′C′ are both isosceles,

the $\angle^s$ OBC, OCB, O′B′C′, O′C′B′ are all equal [*Th* 16]

$\triangle^s$ OBC, O′B′C′ are equiangular to one another,

$R : R' = OB : O'B' = BC : B'C'$

= ratio of a pair of corresponding sides

(iii) Let I, I′ be the two in-centres

Then $\angle IBC = \frac{1}{2}\angle ABC = \frac{1}{2}\angle A'B'C' = \angle I'B'C'$

Similarly $\angle ICB = \angle I'C'B'$,

the $\triangle^s$ ICB, I′C′B′ are equiangular to one another, [*Th* 16]

Draw IK, I′K′ perp to BC, B′C′

by (i), $IK : I'K' = BC : B'C'$

That is $r : r' = BC : B'C'$

3 Let D, E, F be the mid-pts of the sides BC, CA, AB

Then, by Ex 1 p 263, FE is par[l] to BC, and $= \frac{1}{2}BC$

Similarly $FD = \frac{1}{2}AC$, and $DE = \frac{1}{2}AB$

$\angle EDF = \angle A$, $\angle DEF = \angle B$, and $\angle DFE = \angle C$, [*Th.* 63]

$\triangle^s$ DEF, ABC are similar to one another

ratio of circum-radii = ratio of corresponding sides [*Ex* 2]

$= EF : BC = 1 : 2$

4 Join AD, BC Then in the $\triangle^s$ AXD, CXB, we have

$\angle AXD = \angle CXB$, and $XA : XC = XD : XB$,

by Theor 64, the $\triangle^s$ are similar

$\angle XBC = \angle XDA$, or, $\angle ABC = \angle ADC$

A, D, B, C are concyclic [*Th* 39 *Converse*]

5 Join AP, AQ In $\triangle^s$ ABP, ACQ we have $\angle ABP = \angle ACQ$ [*Th* 14],

and $AB : AC = BP : CQ$, the $\triangle^s$ are similar [*Th* 64]

$\angle BAP = \angle CAQ$, A, P, Q are collinear

6. See Theor. 65.

(i) If $c<b$, then $c'<b'$, since $\frac{c}{b}=\frac{c'}{b'}$:

∴ C<B, and C<B'; but B=B;

∴ C, C' are each <B'.

∴ C, C are both *acute*.

∴ C, C' cannot be supplementary, and must ∴ be equal to one another: hence the ∠ˢ are similar.

In (ii), as in (i), the ∠ˢ are similar.

In (iii) the ∠ˢ C C' may be *either* equal or supplementary as in Figs. 1 and 3 of Theor. 65.

7. Because ∠ˢ RPQ, RAB are equiangular to one another,

∴ PQ : AB=PR : AR [*Th.* 62.]

And because ∠ˢ SPQ, SDC are equiangular to one another,

∴ PQ : DC=PS : DS.

But AB=DC; ∴ PR : AR=PS : DS.

∴ SR is parˡ to AD. [*Th.* 60.]

8. Because ∠ˢ ABD, AEC are in same segment, they are equal. And ∠BAD=∠EAC ∴ ∠ˢ ABD, AEC are equiangular. [*Th.* 16.]

∴ AB : AE=AD : AC [*Th.* 62.]

∴ AB . AC=AE . AD

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1. In the ∠ˢ PQA, PAB

the ∠PQA=30°=∠PAB; and the ∠P is common;

∴ the third ∠ˢ are equal [*Th.* 16], and the ∠ˢ are equiangular:

∴ PQ : PA=PA : PB; [*Th.* 62.]

∴ PA^2=PQ . PB [III. p. 250.]

$=3a\times a=3a^2$.

2. By Theor. 29, $BC=\sqrt{AB^2+AC^2}=\sqrt{4^2+3^2}=5{\cdot}0''$.

Also BA^2=BD . BC [*Th.* 66, Cor. (ii)].

∴ $BD=\frac{BA^2}{BC}=\frac{16}{5}=3{\cdot}2''$: and DC=BC−BD=1·8''.

3 (i) Area of rt $\angle^d$ $\triangle$ ABC $=\frac{1}{2}$AB AC [*Th* 25]

Also $\triangle$ ABC $=\frac{1}{2}$ BC AD [*Th* 25]

AB AC $=$ BC AD

(ii) The $\triangle^s$ BCA, ACD are similar ; [*Th.* 66]

BC AC $=$ BA AD [*Th* 62]

BC AD $=$ AB AC.

4 By Theor 29, BC $=25$ cm

By Theor 66, BC′ BC $=$ BA^2, BC′ $=16$ cm, and C′C $=9$ cm

By Theor 66, AC′^{2} $=$ BC′ C′C whence AC′ $=12$ cm

By Theor 62, C′A′ CA $=$ C′B CB, whence C′A′ $=9\ 6$ cm

5 (i) See Ex 9, p 177

(ii) The radius CP is perp to tangent QR ,

RP PQ $=$ CP^2 [*Th* 66, *Cor* 1]

or, PR PQ $=r^2$

6. (i) See Ex 9, p 187

(ii) Since PAQ is a rt $\angle$, the $\angle^s$ PAY, QAX are both rt $\angle^s$,

PY, QX are diameters

Now $\angle$ PYA $=$ comp^t of $\angle$ APY $=\angle$ QPA, [*Th* 46]

and the rt $\angle$ YPQ $=$ rt $\angle$ PQX,

$\triangle^s$ YPQ, PQX are similar, [*Th* 62]

YP PQ $=$ PQ QX,

PQ^2 $=$ YP QX $=2r\ \ 2r'=4rr'$

7. Let C be the centre of $\bigcirc$ on which P lies Join CP

Then since $\angle$ PAQ is a rt $\angle$ [*Ex* 6],

the $\angle$ SAQ $=$ comp^t of $\angle$ CAP $=$ comp^t of $\angle$ CPA $=\angle$ APS,

and $\angle$ at S is common to the two $\triangle^s$ SAP, SQA

the third $\angle^s$ are equal [*Th* 16], and the $\triangle^s$ are equiangular,

SP SA $=$ SA SQ, *i e* SP SQ $=$ SA^2

8 Because AD touches $\bigcirc$ ACB at A, $\angle$ BAD $=\angle$ ACB, [*Th* 49]
and because AC touches $\bigcirc$ ADB at A, $\angle$ BAC $=\angle$ BDA

$\triangle^s$ BCA, BAD are equiangular to one another, [*Th* 16]

BC BA $=$ BA BD [*Th* 62]

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1 By Theor 29, $c=17$

2 Let $a=35$, $b=12$, then by Theor 29, $c=\sqrt{12^2+35^2}=37$
B is the smallest angle

3 Let ABC be a $\triangle$, rt $\angle^d$ at C

Then $$BC^2+CA^2=AB^2 \qquad [Th. 29]$$

(i) $$\frac{BC^2}{AB^2}+\frac{CA^2}{AB^2}=1,$$

or $$\sin^2 A+\cos^2 A=1.$$

And (ii) $$\frac{BC^2}{CA^2}+1=\frac{AB^2}{CA^2},$$

or $$\tan^2 A+1=\sec^2 A.$$

4. By Theor 29, $BC=\sqrt{(1\cdot5)^2+(3\cdot6)^2}=\sqrt{15\cdot21}=3\cdot9$ cm

and $CD=\sqrt{(8\cdot5)^2-(3\cdot6)^2}=\sqrt{12\cdot1\times49}=7\cdot7$ cm

$$\therefore \sin ABC=\frac{AC}{CB}=\frac{3\cdot6}{3\cdot9}=\frac{12}{13},$$

$$\tan ACB=\frac{AB}{AC}=\frac{1\cdot5}{3\cdot6}=\frac{5}{12},$$

$$\cos CDA=\frac{CD}{DA}=\frac{7\cdot7}{8\cdot5}=\frac{77}{85};$$

$$\tan DAC=\frac{DC}{CA}=\frac{7\cdot7}{3\cdot6}=\frac{77}{36}$$

5 In the Fig on p 270,

$$\angle ABC=\text{comp}^t \text{ of } A=90^\circ-A,$$

$$\therefore \sin(90^\circ-A)=\sin ABC=\frac{AC}{AB}=\cos A,$$

$$\tan(90^\circ-A)=\tan ABC=\frac{AC}{CB}=\cot A.$$

6 The ratio of the opposite side to the hypotenuse is 6 : 10

Construct a rt $\angle^d$ $\triangle$, having hypotenuse $=10$ cm and one side $=6$ cm The angle opposite this latter side is the req^d angle

7 (i) Draw AC=10 cm, CB perp to AC and=7 cm

Then BAC is the req^d $\angle$

(ii) Construct a rt $\angle^d$ $\triangle$ whose hypotenuse=10 cm, and having one side=9 cm, the angle included by these sides is the req^d $\angle$ [See Prob 10]

(iii) Construct a rt $\angle^d$ $\triangle$, with hypotenuse=10 cm, and one side=7 1 cm The angle opp this side is the req^d $\angle$

8 Draw AC=5 cm, CB perp to AC and =8 cm, then

$$\tan BAC=\frac{BC}{CA}=\frac{8}{5}=1\ 6$$

By Theor 29, $AB^2=64+25$, whence $AB=\sqrt{89}$

$$\cos BAC=\frac{AC}{AB}=\frac{5}{\sqrt{89}}=\frac{5\sqrt{89}}{89}=\frac{5\times 9\ 434}{89}=0\ 53$$

9 (i) Draw a rt-$\angle^d$ isosceles $\triangle$ ABC, C being the rt $\angle$

Then by Theor 16, $\angle BAC=\angle ABC=45°$

Let AC=a units, CB=a, and AB=$a\sqrt{2}$ [*Ex* 11, p 123]

$$\sin 45°=\sin A=\frac{BC}{AB}=\frac{a}{a\sqrt{2}}=\frac{1}{\sqrt{2}},\quad \cos 45°=\frac{AC}{AB}=\frac{a}{a\sqrt{2}}=\frac{1}{\sqrt{2}}$$

(ii) Draw an equilateral $\triangle$ ABC

Bisect $\angle$ABC by BD meeting AC in D Then, by Theor 4, AD=DC, and $\angle$BDA=rt $\angle$

Let AB=$2a$ units, AD=a, and BD=$a\sqrt{3}$ [*Ex* 14, p 124]

$$\sin 60°=\sin BAD=\frac{BD}{BA}=\frac{a\sqrt{3}}{2a}=\frac{\sqrt{3}}{2};$$

$$\text{and }\cos 30°=\cos ABD=\frac{BD}{BA}=\frac{a\sqrt{3}}{2a}=\frac{\sqrt{3}}{2}$$

10 Draw AD=10 cm, DE perp to AD and=8 1 cm

Join AE, and cut off AB along AE=10 cm Draw BC perp to AD

Then $$\tan A=\frac{BC}{CA}=\frac{ED}{DA}=\frac{8\ 1}{10}=0\ 81$$

By measurement, AC=7 8 cm, BC=6 3 cm, and $\angle$A=39°

$$\therefore \quad \sin A=\frac{BC}{AB}=\frac{6\ 3}{10}=0\ 63,$$

$$\text{and } \cos A=\frac{AC}{AB}=\frac{7\ 8}{10}=0\ 78$$

11. Draw AD=10 cm DE perp to AD and=7 cm Join AE Along AD mark off AC=2 8 cm, and through C draw CB par[l] to DE and meeting AE in B

Then $$\tan A=\frac{BC}{CA}=\frac{ED}{DA}=\frac{7}{10}=0\ 7$$

By measurement, c=AB=3 4 cm., A=35°.

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2. If two $\triangle$s ABC, A'B'C' are equiangular, we have

$$\sin A=\sin A', \quad \sin B=\sin B', \quad \sin C=\sin C'$$

But $a \ b=\sin A \ \sin B$, and $a' \ b'=\sin A' \ \sin B'$;

$$. \ a \ b=a' \cdot b',$$

$$\text{or } a . a'=b \ b',$$

$$=c \ c', \text{ similarly}$$

4. (1) Let ABCD be the par^{m}.

Then area=2$\triangle$ADB=AB AD sin DAB [*Ex.* 3]

(2) Let BAD be the given $\angle$ of the rhombus

Then by the above, area=AB.AD sin DAB=AB^2 sin DAB

5 Let BD be the diameter through B, join DC

Then $\angle$BDC=$\angle$BAC in the same segment [*Th* 39]

Also $\angle$BCD is a rt $\angle$ [*Th* 41]

$$\sin A=\sin BDC=\frac{BC}{BD}=\frac{a}{2R}$$

$$\therefore \ R=\frac{a}{2\sin A}=\frac{a}{2 \ \frac{2\Delta}{bc}} \text{ (by Ex. 3)}, =\frac{abc}{4\Delta}$$

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3 (i) In Prob 35 make DG=1 0″, GE=1 6″, DH=1 25″, the length of HF gives the req^d value of x

(ii) Here $6\,3 \quad 4\,2 \quad 4\,2 \quad x$

Take the *third* proportional to 6 3 cm and 4 2 cm [*Prob* 36]

(iii) Here $16 \quad x \quad x \quad 25$,

take the *mean* proportional between 1 6″ and 2 5″ [*Prob* 38]

4. In Cor to Prob 37, make AB=7 2 cm, and take AP=2 cm, PQ=3 cm, QR=4 cm

5 The third part$=\frac{2}{4}\times\frac{2}{3}$ of the first$=\frac{1}{2}$ of the first

the three parts are in the ratio of 1 $\frac{2}{3}$ $\frac{1}{2}$ or 6 4 3

Now proceed as in Prob 37 Cor

6 If x be the other side of the rect we have

$$1\,5\times x=2^2, \text{ or } 1\,5 \quad 2=2 \quad x$$

Hence find the *third* proportional to 1 5 and 2 0 [*Prob* 36]

7 Find *mean* proportionals between

(i) 3 0″ and 1 0″ (ii) 5 0 cm and 2 0 cm

(iii) 2 0″ and 1 4″ [See p 277]

8. (i) If $x=\dfrac{3\,5\times 2\,4}{2\,8}$, then $2\,8 \quad 3\,5=2\,4 \quad x$,

hence find the 4^th proportional to 2 8″, 3 5″, 2 4″ [*Prob* 36]

(ii) $\dfrac{6\,84}{2\,13}=\dfrac{68\,4}{21\,3}=\dfrac{22\,8}{7\,1}=\dfrac{4\,0\times 5\,7}{7\,1}=x$

Then $x=$the *fourth* proportional to 7 1 cm, 4·0 cm, 5 7 cm

(iii) Find the *fourth* proportional to 1 51″, 2 71″, 1 26″

9. (i) Draw an acute ∠KBM, making BK=4 8″ Along BM cut off BP, PQ, QR=3, 4, 5 units respectively

Join RK, and through P, Q draw par[ls] to RK meeting BK in C and D

With centres B, C and radii DK, CD respectively, describe arcs cutting at A Then ABC is the required △

By calculation, $a=\frac{3}{3+4+5}$ of 4 8″=1 2″, and similarly for b and c

(ii) Here $a=\frac{5}{6}b, \quad c=\frac{5}{4}b,$

$$a \quad b \quad c=\tfrac{5}{6} \quad 1 \quad \tfrac{5}{4}=2 \quad 24 \quad 3$$

Now proceed as in (i)

(iii) By Theor 16, A=30°, B=60°, C=90°

Draw *any* △ RBQ having B=60°, Q=90°, R=30°

Produce BR to K so that BK=perimeter of △ BRQ

Along BK mark off BP=11 8 cm, join KQ, and through P draw PC par[l] to KQ meeting BQ in C Through C draw CA par[l] to QR to meet BK in A Then ABC is the req[d] △

By parallels, the △[s] ABC, RBQ are equiangular,

BA BR=AC RQ=CB QB=BA+AC+CB BR+RQ+QB [*Th* V p 251]

perimeter of △ ABC perimeter of △ RBQ=BC BQ,
=BP BK

But the perimeter of △ RBQ=BK

perimeter of △ ABC=BP=11 8 cm

Or by calculation thus

As in Ex 14, p 124,

$$\frac{a}{1}=\frac{b}{\sqrt{3}}=\frac{c}{2}=\frac{a+b+c}{1+\sqrt{3}+2} \qquad [Th\ V, p\ 251]$$

$$a=\frac{1}{3+\sqrt{3}} \text{ of } 11\ 8 \text{ cm} =\frac{(3-\sqrt{3})11\ 8}{6} \text{ cm} =\frac{11\ 8\times 1\ 268}{6} \text{ cm}$$

$$=2\ 5 \text{ cm, nearly}$$

(iv) Draw PA=5 0″, AQ perp to PA and=3 0″, join PQ.
Along PQ mark off PR=4 0′, through R draw RB par[1] to PA and meeting AQ in B Through B draw BC par[1] to PQ

Then, by parallels, the Δ^s ABC, AQP are equiangular,

$$AC \quad AB = AP \quad AQ = 5 \quad 3$$

By calculation, let $b=5k$, $c=3k$,

by Theor 29, $(5k)^2+(3k)^2=16$,

$$k=\frac{4}{\sqrt{34}}=\frac{2\sqrt{34}}{17}=\frac{11\ 66}{17}=0\ 686$$

$b=3\ 4''$, $c=2\ 1''$, nearly

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1 (i) Here $\dfrac{a}{8\ 0}=\dfrac{b}{5\ 6}=\dfrac{c}{6\ 4}$, and $a=200$ metres

$$b=\frac{5\ 6}{8\ 0} \text{ of } 200 \text{ m}, \text{ and } c=\frac{6\ 4}{8\ 0} \text{ of } 200 \text{ m}$$

(ii) PQ BC=AP AB=4 0 6 4

$$\text{length of fence}=\frac{4\ 0}{6\ 4} \text{ of } 200 \text{ m} = 125 \text{ m}$$

2 Draw any angle POQ Cut off OQ 10 cm in length to represent 100 yds on a scale of 10 yds to 1 cm Along OP mark off OA=8 cm and OB=7 cm to represent the distances reached by A and B at the end of any interval Join AQ and through B draw BK par[1] to AQ to meet OQ in K Then since OQ OK=OA OB=8 7, the distance by which A beats B is measured by KQ which is found to be 1 25 cm Hence A wins by (1 25 × 10) yds, or 12 5 yds

3 On drawing a plan it is seen that BAC is a rt ∠,

$$BC=\sqrt{(0\ 8)^2+(1\ 5)^2}=1\ 7''$$

But 1″ represents 25 mi, distance between B and C

$$=(1\ 7\times 25) \text{ mi} = 42\tfrac{1}{2} \text{ mi}$$

4. Let x ft be the height of the lamp-post Then by similar Δ^s,

$$x \quad 6=40 \quad 8, \text{ whence } x=30$$

Let y ft be the length of the boy's shadow,

$$20+y \quad y=30 \quad 5=6 \quad 1 \qquad 6y=20+y, \text{ and } y=4$$

5. If x ft be the height of lamp-post, we have by similar Δ^s

$$x : 6 = 20 : 5, \text{ whence } x = 24$$

Let y ft be the length of man's shadow in his 2[nd] position

$$7+y : y = 24 : 6 = 4 : 1$$
$$4y = 7+y, \quad \text{or } y = 2\tfrac{1}{3}$$

6 Let AB represent the breadth of the canal, C in AB produced the stand-point of the man, CE, BD the heights of the man's eye and of the rod respectively Let AB $= x$ ft Then by the details of the question, EDA is a st line Hence the Δ^s ABD, ACE are similar

$$AB : AC = BD : CE,$$

or $$x : x+20 = 4\tfrac{1}{4} : 5\tfrac{2}{3} = 3 : 4$$
$$4x = 3(x+20),$$

whence $$x = 60$$

7. Let AB, CD, EF represent the heights of the tower, staff, and man respectively, A, C, E being on the ground Through F draw a horizontal FKL cutting CD in K and AB in L

Then Δ^s FKD, FLB are similar

$$LB : KD = LF : KF,$$

or $$LB : 6\tfrac{2}{3} = 30 : 3$$
$$LB = 66\tfrac{2}{3} \text{ ft}$$

Now height of tower $=$ AL $+$ LB $= 72$ ft

8 For the ground plan let A represent the foot of the light-house, C, D the man's two stand-points, P and Q the extremities of the shadow in the two positions

Then $\angle$ACD $=$ rt $\angle$, CP $= 24$ ft, DQ $= 30$ ft

Let AB, a vertical at A, represent the light-house, x ft high, and CE, DF the man's height in his two positions Then CE, DF both represent 6 ft, and BEP, BFQ are st lines

From similar Δ^s BAP, ECP, we have AP : CP $=$ AB : CE,

or, AP : 24 $= x$: 6, whence AP $= 4x$, and AC $= 4x - 24$

Similarly from Δ^s BAQ, FDQ, AQ $= 5x$, and AD $= 5x - 30$

From the rt-$\angle^d$ Δ ACD, we have $AD^2 = AC^2 + CD^2$,

or, $(5x-30)^2 = (4x-24)^2 + 300^2$, since CD $= 300$ ft

$$25(x-6)^2 = 16(x-6)^2 + 90000,$$

whence $$x = 106$$

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1 (i) In the Fig of Theor 68 we see that if A′B′ AB=3 4, we have also SA′ SA=3 4 Hence the following construction

Take any pt S as centre of similarity, and join SA, SB, SC, SD

Divide SA internally at A′ so that SA′ SA=3 4

Through A′ draw A′B′ par[l] to AB to meet SB in B′

Through B′ draw B′C′ par[l] to BC to meet SC in C′

Through C′ draw C′D′ par[l] to CD to meet SD in D′

Join A′D′ Then A′B′C′D′ is the req[d] quad[l]

From the similar $\triangle^s$ SAB, SA′B′, SA′ SA=SB′ SB

Similarly SB′ SB=SC′ SC, and SC′ SC=SD′ SD

SA′ SA=SD′ SD, whence A′D′ is par[l] to AD [*Th* 60]

Hence A′B′C′D′ has each of its sides par[l] to the corresponding side of ABCD, and is equiangular to it

Also A′B′ AB=SA′ SA=3 4

(ii) Divide SA externally at A′ so that SA′ SA=5 4, and proceed as in (i)

2 On AB construct the square ABCD Join O (the centre of the semi-○) to C, D by lines cutting the ○[ce] in c, d Through c, d draw cb, da par[l] to CB and meeting AB in b and a Join cd

Then Oc=Od, and OC=OD, whence cd is par[l] to CD Thus $abcd$ is a rectangle

Also by similar $\triangle^s$, cb CB=Oc OC=cd CD

And CB=CD, $cb=cd$ Hence $abcd$ is a square

It is easy to prove Oa=Ob O$b=\frac{1}{2}a$

But $Oc^2=Ob^2+bc^2$ $r^2=\dfrac{a^2}{4}+a^2$, whence $4r^2=5a^2$

3 Let OA, OB be the bounding radii of the sector On AB, and on the side of it remote from O, describe the sq ABCD Join OC, OD cutting the arc in c and d Join cd Then it can easily be shewn that cd is par[l] to CD From c and d draw cb, da par[l] to CB, DA Join ab

As in Ex 1, it may be shewn that ab is par¹ to AB

Also by similar $\triangle^s$, $ab : AB = Oa : OA = ad : AD$,

∴, *alternately*, $ab : ad = AB : AD$,

but $AB = AD$, ∴ $ab = ad$, and $abcd$ is a square.

By measurement, $a = 1\cdot24''$, whence $a : r = 0\cdot52 : 1$

4. Let OA, OB be the bounding radii of the sector

On the chord AB describe a rect ABCD having $AD = 2AB$

Complete the construction as in Ex 3 and prove ab par¹ to cd

Then $ab : AB = Oa : OA = ad : AD$

∴ *alternately*, $ab : ad = AB : AD = 1 : 2$

Thus $abcd$ is a rect having $ad = 2ab$

By describing on AB a rect such that $AD = \frac{1}{2}AB$, we can inscribe in the sector another rect $a'b'c'd'$ having $a'd' = \frac{1}{2}a'b'$

By measurement, $ad = 3\cdot1$ cm, $a'b' = 2\cdot8$ cm

$ad : a'b' = 31 : 28$

5. On the side of BC remote from A describe the square BCDF.

Join AD, AF cutting BC in d and f

Through d, f draw dc and fb par¹ to DC, FB to meet AC, AB in c and b Join bc.

As in Ex 1, bc is par¹ to BC, hence $bcdf$ is a rect

Also by similar $\triangle^s$, $bc : BC = Ac : AC = cd : CD$,

alternately, $bc : cd = BC : CD$,

but $BC = CD$, $bc = cd$, and $bcdf$ is a square

6. (i) On the side of BC remote from A describe an equilateral $\triangle$ DBC

Join AD cutting BC in d From d draw db and dc par¹ to DB, DC and meeting AB, AC in b and c Join bc

Then by similar $\triangle^s$, $Ab : AB = Ad : AD = Ac : AC$

Thus bc is par¹ to BC

Hence $\triangle dbc$ has each of its sides par¹ to the corresponding side of $\triangle$ DBC, and is equiangular to it

That is, dbc is an equilateral $\triangle$

(ii) Draw RQ par[l] to the given line to meet AB, AC in R and Q On QR describe the equilateral △ PQR

Join AP, producing it if necessary to cut BC in p From p draw pq, pr par[l] to PQ, PR, and meeting AC, AB in q and r

Then, as in Ex 6 (i), the △ pqr is equilateral

Draw any line RQ cutting AB, AC in R and Q.

On the side of QR remote from A describe a △ PQR equiangular to the △ DEF

Join AP, producing it if necessary to cut BC in p Through p draw pq, pr par[l] to PQ, PR and meeting AC, AB in q and r

Then, as in Ex 6, the △ pqr is equiangular to the △ PQR, and also to the △ DEF

Hence, by Theor 64, the △ pqr is similar to the △ DEF

Since RQ may be drawn in any direction, the number of solutions is infinite

8 On the same side of DE as AB describe the square DEMK

Produce CD, FE to meet in O, and let OK, OM cut BC, AF in P and Q.

Join PQ, and draw PS, QR par[l] to KD, ME and meeting CD, EF in S and R Join RS Then PQRS is the req[d] square

It is easy to shew that △[s] OED, OFC are isosceles

Hence from the congruent △[s] ODK, OEM, we have OK=OM, and also ∠ DOK=∠ EOM

by Theor 17 the △[s] OCP, OFQ are congruent, and OP=OQ.

Hence PQ is par[l] to KM

Next, as in Ex 1, prove DE par[l] to SR Then PQRS is a par[m], and is rectangular by construction

Also by similar △[s], PQ : KM=OP : OK=PS : KD

Alternately, PQ : PS=KM : KD

But KM=KD, PQ=PS, and the figure is a square

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1. By Theor. 70, $\Delta : 12\frac{1}{4}$ sq. in. $= 5\cdot4'' : 6\cdot3''$

$$\Delta = \frac{5\cdot4}{6\cdot3} \text{ of } 12\tfrac{1}{4} \text{ sq. in.} = 10\tfrac{1}{2} \text{ sq. in.}$$

2. Req^d base : $4\cdot2$ cm $= 17 : 24$

$$\text{base} = \tfrac{17}{24} \text{ of } 4\cdot2 \text{ cm} = 2\cdot975 \text{ cm}$$
$$= 3\cdot0 \text{ cm, to the nearest mm.}$$

3. $\Delta : 50\cdot1204$ sq. metres $= 20\cdot7$ m : $16\cdot2$ m

$$\Delta = \tfrac{207}{162} \text{ of } 50\cdot1204 \text{ sq. metres},$$
$$= 64\cdot0427 \text{ sq. metres, to nearest sq. mm.}$$

4. Req^d base : $6\cdot6'' = 35 : 21$ [*Th.* 70, *Cor.*]

$$\text{base} = \tfrac{35}{21} \text{ of } 6\cdot6'' = 11\cdot0''$$

5. Let Δ_1, Δ_2 denote the areas of the two triangles, a their common base. $\Delta_1 = \frac{1}{2} a \times 4\cdot2$, and $\Delta_2 = \frac{1}{2} a \times 3\cdot71$

$$\Delta_2 : \Delta_1 = 3\cdot71 : 4\cdot2,$$
$$\Delta_1 + \Delta_2 : \Delta_1 = 7\cdot91 : 4\cdot2, \qquad [\textit{Componendo.}]$$
$$\Delta_1 + \Delta_2 = \frac{7\cdot91}{4\cdot2} \text{ of } \Delta_1 = \frac{113}{60} \text{ of } 18 \text{ acres}$$
$$= 33\cdot9 \text{ acres}$$

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1. (i) Let Δ_1, Δ_2 denote the areas of the two Δ^s, a their common base, p_1, p_2 their altitudes. $\Delta_1 = \frac{1}{2} a \times p_1$, and $\Delta_2 = \frac{1}{2} a \times p_2$

$$\frac{\Delta_1}{\Delta_2} = \frac{\frac{1}{2} a \times p_1}{\frac{1}{2} a \times p_2} = \frac{p_1}{p_2}$$

(ii) Place the two Δ^s ABC, DBC on same base BC. Through C draw CK perp. to BC, and through A, D draw parallels to BC meeting CK in P and Q. Join PB, QB.

Then ΔABC $= \Delta$PBC, and ΔDBC $= \Delta$QBC [*Th.* 26.]

ΔABC : ΔDBC $= \Delta$PBC : ΔQBC,

= base PC : base QC,

(for the Δ^s PBC, QBC have same altitude BC)

= alt. of ΔABC : alt. of ΔDBC

2 (i) Here $\triangle BXY = \triangle CXY$ [*Th* 26]

$\triangle AXY \;\; \triangle BXY = \triangle AXY \;\; \triangle CXY$ [*Ax* (ii), p 249]

But $\triangle AXY \;\; \triangle BXY = AX \;\; XB$, [*Th* 70]

and $\triangle AXY \;\; \triangle CXY = AY \;\; YC$ [*Th* 70]

$AX \;\; XB = AY \;\; YC$

(ii) $\triangle BXY = \triangle CXY$ Add to each $\triangle AXY$

$\triangle ABY = \triangle AXC$

$\triangle ABY \;\; \triangle AXY = \triangle AXC \;\; \triangle AXY$

But $\triangle ABY \;\; \triangle AXY = AB \;\; AX$, [*Th* 70]

and $\triangle AXC \;\; \triangle AXY = AC \;\; AY$, [*Th* 70]

$AB \;\; AX = AC \;\; AY$

3 In the quad[l] ABCD let the diag[s] AC, BD intersect in E

Then $\triangle ABE \;\; \triangle EBC = AE \;\; EC$ [*Th* 70]

$= \triangle ADE \;\; \triangle EDC$

4 Let the $\triangle$[s] ABC, DEF be on equal bases BC, EF, and between the same par[ls] AD and BF

Let a par[l] to BF cut AB, AC in X, Y, and DE, DF in P, Q.

Then $XY \;\; BC = AY \;\; AC$, [*Th* 62]

and $PQ \;\; EF = DP \;\; DE$

But $AY \;\; AC = DP \;\; DE$ [*Ex* 4, p 258]

$XY \;\; BC = PQ \;\; EF$

But $BC = EF$ $XY = PQ$ $\triangle AXY = \triangle DPQ$. [*Th* 26]

5 By Theor 71, $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{AB}{DE}\dfrac{BC}{EF}$

$= \dfrac{2\,7 \times 3\,5}{2\,1 \times 1\,8} = \dfrac{5}{2}$

6 Here $\triangle ABC \;\; \triangle DEF = AB \;\; BC \;\; DE \;\; EF$ [*Th* 71]

And $\triangle ABC = \triangle DEF$, $AB \;\; BC = DE \;\; EF$

$5\,6 \times 6\,3 = 7\,2 \times EF$, whence $EF = 4\,9$ cm

7. By Theor. 71. Cor., $\dfrac{\text{par}^{m} ABCD}{\text{par}^{m} EFGH}=\dfrac{AB . BC}{EF . FG}$:

hence by hypothesis $\dfrac{3}{4}=\dfrac{4.8\times13.5}{10.8\times FG}$;

$\therefore 3\times10.8\times FG=4\times4.8\times13.5$;

whence $FG=8.0$ cm.

But the area of par^{m} $ABCD=BC\times p$;

and area of par^{m} $EFGH=FG\times p'$. [*Th.* 24, *Cor.*]

$\therefore \dfrac{13.5\times p}{8.0\times p'}=\dfrac{3}{4}$;

$\therefore \dfrac{p}{p'}=\dfrac{3\times8.0}{4\times13.5}=\dfrac{4}{9}$.

8. See Ex. 3 p 273

$\triangle ABC=\frac{1}{2}AB . BC\sin B$, and $\triangle DEF=\frac{1}{2}DE . EF\sin E$.

But $\angle B=\angle E$. and $\therefore \sin B=\sin E$.

Thus $\dfrac{\triangle ABC}{\triangle DEF}=\dfrac{\frac{1}{2}AB . BC\sin B}{\frac{1}{2}DE . EF\sin E}=\dfrac{AB . BC}{DE . DF}$

or. $\triangle ABC : \triangle DEF=AB . BC : DE . EF$.

9 If the $\angle E$ is not equal to the $\angle B$, at the pt. E draw EK making an $\angle FEK=\angle B$, and cutting a par^{l} to EF through D in K. Join KF.

Then $\triangle KEF=\triangle DEF$ [*Th.* 26]$=\triangle ABC$.

Also $\triangle KEF : \triangle ABC=KE . EF : AB . BC$; [*Th.* 71.]

$\therefore KE . EF=AB . BC$

But $DE . EF=AB . BC$. [*Hyp and Th.* III. p 250.]

$\therefore KE=DE$, and $\therefore \angle EKD=\angle EDK$.

Produce FE to Q; $\therefore \angle KEF=\angle DEQ$. [*Th* 14.]

But $\angle^{s}$ DEF, DEQ are supplementary;

$\therefore$ $\angle^{s}$ DEF, KEF are supplementary;

$\therefore$ $\angle^{s}$ DEF, ABC are supplementary. [*Const*]

10. By Theor. 70, $\dfrac{\triangle APQ}{\triangle APC}=\dfrac{AQ}{AC}$, and $\dfrac{\triangle APC}{\triangle ABC}=\dfrac{AP}{AB}$;

$\therefore$ by multiplication, $\dfrac{\triangle APQ}{\triangle ABC}=\dfrac{AQ . AP}{AC . AB}$

Theor 71 follows by imagining one $\triangle$ to be superposed to the other with the equal angles coinciding, and then applying the above result.

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1 $\triangle^s$ AXY, ABC are equiangular [*Th* 14], and similar [*Th* 62]

$$\triangle AXY \quad \triangle ABC = AX^2 \quad AB^2 \qquad [Th\ 72]$$
$$= 1^2 \quad 3^2 = 1 \quad 9$$

2 $$\text{Req}^d\ \triangle \quad 45 \text{ sq ft} = (2\tfrac{1}{3})^2 \quad (3\tfrac{1}{2})^2$$
$$= 4 \quad 9$$
$$\triangle = \tfrac{4}{9} \text{ of } 45 \text{ sq ft} = 20 \text{ sq ft}$$

3 Here $AX \quad AB = 5 \quad 8$,

and $\triangle AXY \quad \triangle ABC = AX^2 \quad AB^2$ [*Th* 72]

$$= 25 \quad 64$$

$$\triangle AXY = \tfrac{25}{64} \text{ of } \triangle ABC = \tfrac{25}{64} \text{ of } 25\ 6 \text{ sq cm} = 10 \text{ sq cm}$$

4 Areas of the $\triangle^s$ are in ratio 392 200, *i e.* 49 25, or $7^2 \quad 5^2$
any pair of corresponding sides are as 7 5 [*Th* 72]

5 Here $\triangle ABC \quad \triangle XYZ = 32 \quad 60\ 5 = 64 \quad 121 = 8^2 \quad 11^2$

and $\triangle ABC \quad \triangle XYZ = AB^2 \quad XY^2$, [*Th* 72]

$AB \quad XY = 8 \quad 11$,

$AB = \tfrac{8}{11}$ of $XY = \tfrac{8}{11}$ of 7 7″ = 5 6″

6 Here $AX^2 \quad AB^2 = \triangle AXY \quad \triangle ABC = 9 \quad 16$, [*Th* 72]

$AX \quad AB = 3 \quad 4$

Hence divide AB at X in ratio of 3 1

7 The $\triangle^s$ BAD, ACD are similar [*Th* 66], and BA, AC are corresponding sides, $\triangle BAD \quad \triangle ACD = BA^2 \quad AC^2$ [*Th* 72]

8 The $\triangle^s$ AOB, COD are similar [*Th* 14 and 62], and AB, CD are corresponding sides,

$$\triangle AOB \quad \triangle COD = AB^2 \quad CD^2 = 2^2 \quad 1^2 = 4 \quad 1$$

9 Inversely, fig $XBCY \ . \ \triangle AXY = 5 \quad 4$,

, componendo, fig $XBCY + \triangle AXY \quad \triangle AXY = 5+4 \quad 4$,

or, $\triangle ABC \quad \triangle AXY = 9 \quad 4$

$AB^2 \quad AX^2 = 9 \quad 4$, [*Th* 72.]

or, $AB \quad AX = 3 \quad 2$

, dividendo, $AB - AX \quad AX = 3-2 \quad 2$, *i e* $BX \quad AX = 1 \quad 2$,

or, $AX \quad XB = 2 \quad 1$

10. Let ABC, A'B'C' be similar $\triangle^s$ [See *Ex* 2, p 267]

(i) Let AK, A'K' be corresponding altitudes

AB A'B' = AK A'K', from similar $\triangle^s$ABK, A'B'K'

But $\triangle$ABC $\triangle$A'B'C' = AB^2 $A'B'^2$ [*Th* 72]

= AK^2 $A'K'^2$

(ii) Let AD, A'D' be corresponding medians

Then AB A'B' = BD B'D', and $\angle$B = $\angle$B'

$\triangle^s$ABD, A'B'D' are similar [*Th* 64], and AD, A'D' are corresponding sides, AB A'B' = AD A'D'

But $\triangle$ABC $\triangle$A'B'C' = AB^2 $A'B'^2$

= AD^2 $A'D'^2$

(iii) Let IX, I'X' be corresponding in-radii perp to BC, B'C'

Then IX I'X' = IB I'B', from similar $\triangle^s$IBX, I'B'X',

= BC B'C', from similar $\triangle^s$IBC, I'B'C'

But $\triangle$ABC $\triangle$A'B'C' = BC^2 $B'C'^2$,

= IX^2 $I'X'^2$

(iv) Let S, S' be the respective circum-centres

Then SB S'B' = BC B'C', from similar $\triangle^s$SBC, S'B'C'

But $\triangle$ABC $\triangle$A'B'C' = BC^2 $B'C'^2$

= SB^2 $S'B'^2$

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1 Here $$\frac{AX^2}{AB^2} = \frac{\triangle AXY}{\triangle ABC} = \frac{4}{9} = \left(\frac{2}{3}\right)^2$$ [*Th* 72]

AX must be two-thirds of AB

2 Each side of the required $\triangle$ must be $\sqrt{3}$ times the corresponding side of the given $\triangle$ [N B $\sqrt{3} = 1\ 732$]

3 Similar $\triangle^s$ have the same ratio as the squares of their corresponding altitudes (*Ex* 10, p 291)

ratio of altitudes = $\sqrt{13\ 69}$ $\sqrt{16\ 81} = 3\ 7$ $4\ 1$

req^d altitude = $\frac{37}{41}$ of $10\frac{1}{4}$ ft = $9\frac{1}{4}$ ft

4 Here $\dfrac{\triangle BXY}{\triangle BAY}=\dfrac{BX}{BA}=\dfrac{5}{8}$, [*Th* 70]

and $\dfrac{\triangle BAY}{\triangle BAC}=\dfrac{AY}{AC}=\dfrac{3}{8}$, [*Th* 70]

, by multiplication, $\dfrac{\triangle BXY}{\triangle ABC}=\dfrac{15}{64}$,

$\triangle BXY=\frac{15}{64}$ of $\triangle ABC=\frac{15}{64}$ of 16 sq cm $=3\ 75$ sq cm

5 Here $\dfrac{XY^2}{BC^2}=\dfrac{\triangle AXY}{\triangle ABC}=\dfrac{1}{5}$, [*Th* 72]

$\dfrac{XY}{BC}=\dfrac{1}{\sqrt{5}}=\dfrac{\sqrt{5}}{5}$

$XY=\dfrac{\sqrt{5}}{5}$ of 10 cm $=2\times 2\ 236$ cm $=4\ 5$ cm (to nearest mm)

6 Area of req^d pentagon area of given pentagon$=(3\ 0)^2\ \ (2\ 5)^2$ [*Th* 73]

req^d area$=\dfrac{30^2}{25^2}$ of $\dfrac{43}{4}$ sq in $=15\ 48$ sq in

7 Since ratio of areas is 1 9, the ratio of corresponding sides is 1 3 [*Th* 73]

length of second rectangle$=\frac{1}{3}$ of 10 8 metres$=3$ 6 metres

breadth of second rectangle$=\frac{5}{12}$ of 3 6 metres$=1$ 5 metres

8 (i) Here 1 inch represents 66 yards,

1 *square* in $(66)^2$ *square* yards,

100 sq in $(66)^2\times 100$ sq yds

$=\dfrac{66\times 66\times 100}{4840}$ acres$=90$ acres

(ii) The shapes of the field and its plan may be regarded as similar polygons Hence their areas are as the squares of corresponding sides, a ratio which is unaffected by the precise shape of the polygons

9 Area of plan$=\frac{1}{2}\times 20\times(24+26)$ sq in $=500$ sq in [*Th* 28]

Here 25^2 sq in represent 1 sq mile

area of field$=\dfrac{500}{25\times 25}$ sq miles$=\frac{4}{5}$ of 640 acres$=512$ acres

10 As in Ex 2, p 111, the area of plan $=84$ sq cm

84 sq cm represent 18900 sq metres

1 sq cm . 225 sq metres

1 cm . $\sqrt{225}$, or 15, metres

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1 (i) By Theor 66, the $\triangle^s$ BCA, BAD are similar;

$$BC\ \ BA = BA\ \ BD,$$

$$BA^2 = BC\ \ BD$$

Hence $\dfrac{BC^2}{BA^2} = \dfrac{BC^2}{BC\ \ BD} = \dfrac{BC}{BD}$

or, $$BC^2\ \ BA^2 = BC\ \ BD$$

(ii) Similarly from $\triangle^s$ BCA, ACD, $BC^2\ \ CA^2 = BC\ \ CD$.

By (i) we have $\dfrac{AB^2}{BC^2} = \dfrac{BD}{BC}$, and by (ii) $\dfrac{AC^2}{BC^2} = \dfrac{CD}{BC}$,

, by addition, $\dfrac{AB^2 + AC^2}{BC^2} = \dfrac{BD + CD}{BC} = 1$,

or $AB^2 + AC^2 = BC^2$

2 By Theor 72, $AX^2\ \ AB^2 = \triangle AXY\ \ \triangle ABC$

$$= 1\ \ 2,$$

$$AX\ \ AB = 1\ \ \sqrt{2}$$

Bisect AB at D, and on AB draw a semi-circle

Draw Dd perp to AB to cut the $\bigcirc^{ce}$ at d

From centre A with radius Ad cut AB at X

Then, as in Note, p 276,

AX is a *mean* proportional between AD, AB

So that $AX^2 = AD\ \ AB = \frac{1}{2}AB^2$

$$\frac{AX^2}{AB^2} = \frac{1}{2}, \text{ hence } \frac{AX}{AB} = \frac{1}{\sqrt{2}}$$

Now through X draw XY par[l] to the base BC

3. Use Theor 70, or Ex 7, p 269 with Theor 73, Cor

4 The $\angle BAD = \angle ACB$ in the alternate segment,

and $\angle CAB = \angle ADB$ in the alternate segment,

$\angle ABC = \angle ABD$ [*Th* 16]

$\triangle ABC \quad \triangle ABD = AB \quad BC \quad AB \quad BD$ [*Th* 71]

$= BC \quad BD$

Or Theor 72 might be used, the $\triangle^s CBA, ABD$ being similar

5. The $\triangle^s DBF, ABC$ are similar, DB and AB being corresponding sides [II, *Cor* ii p 208]

$$\triangle ABC \quad \triangle DBF = AB^2 \quad DB^2$$

$\triangle ABC - \triangle DBF \quad \triangle DBF = AB^2 - DB^2 \quad DB^2$ [*dividendo*],

or, quad^l $AFDC \quad \triangle DBF = AD^2 \quad DB^2$ [*Th* 29]

6 By repeated applications of Ex 1, p 263, in the *fourth* $\triangle XYZ$ the side YZ which corresponds to BC is found to be equal to one-eighth of BC

Now $\triangle XYZ \quad \triangle ABC = YZ^2 \quad BC^2$ [*Th* 72] $= 1 \quad 64$

7 Let ABCDEF be the regular hexagon, P, Q the mid-pts of DC and DE respectively

Now by the usual method of constructing a regular hexagon, $EB = 2a$ and $\angle ECB = 90°$ [*Th* 41]

$$CE^2 = BE^2 - BC^2 = 4a^2 - a^2 = 3a^2$$

Again by Ex 1, p 263, $PQ = \frac{1}{2}CE$, $\quad PQ^2 = \frac{3}{4}a^2$

The inscribed hexagon can be easily proved regular, and similar to original hexagon

area of *second* hexagon area of *first* hexagon

$$= PQ^2 \quad DE^2 = 3 \quad 4$$

Similarly, area of *third* hexagon

$$= \frac{3}{4} \text{ area of } \textit{second} \text{ hexagon}$$

$$= \left(\frac{3}{4}\right)^2 \text{ area of } \textit{first} \text{ hexagon}$$

area of *fifth* hexagon $= \left(\frac{3}{4}\right)^4$ area of *first* hexagon

first hexagon *fifth* hexagon $= 4^4 \quad 3^4 = 256 \quad 81$

8 Let AB, FG be corresponding sides of the similar cyclic polygons ABCDE, FGHKL

Let AM, FN be the diameters of the circum-$\odot^s$ through A, F respectively Join AC, BM, and FH, GN

$\angle$AMB $=\angle$ACB in same segment $=\angle$FHG, [*Th* 67]

$=\angle$FNG in same segment

Also $\angle$ABM $=\angle$FGN, being rt $\angle^s$

$\triangle^s$ABM, FGN are similar, and AB FG $=$ AM FN

But polygon ABCDE polygon FGHKL $=$ AB^2 FG^2, [*Th* 73]

$=$ AM^2 FN^2

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1. For (i) and (ii) see Corollaries of Theor 66

Then $BC^2 = BC \quad BD + BC \quad DC$ [*Th* 50]

$= BA^2 + AC^2$

2 The fig Q fig P $= AC^2 \quad BC^2$ [*Th* 73]

$= \triangle ADC \quad \triangle ABC$ [*Th* 66, 72]

But fig P $= \triangle$ABC, fig Q $= \triangle$ADC

Similarly fig R $= \triangle$ADB

3 The fig Q fig R $= AC^2 \quad AB^2$ [*Th* 73] $=$ 25 64

Also fig Q $+$ fig R $=$ fig P [*Th* 74] $=$ 8 9 sq cm

fig Q $=\frac{25}{25+64}$ of 8 9 sq cm $=$ 2 5 sq cm,

and fig R $=\frac{64}{25+64}$ of 8 9 sq cm $=$ 6 4 sq cm

4 Since the $\angle^s$BGC, YGZ are equal,

$\triangle$BGC $\triangle$YGZ $=$ BG GC YG GZ [*Th* 71]

But BG $=$ 2YG, and GC $=$ 2GZ [p 97]

$\triangle$BGC $\triangle$YGZ $=$ 4GY GZ YG GZ $=$ 4 1

5 Here $\triangle$ABC $\triangle$ADE $=$ AB AC AD AE, [*Th* 71]

AB AC $=$ AD AE (for $\triangle$ABC $=\triangle$ADE), [*Hyp*]

or, 3 6 $\times$ 3 6 $=$ 1 8 $\times$ AE, whence AE $=$ 7 2″

6 In the Fig taken, AP lies between AB and AQ. Join PQ, BQ.

(i) Since $\angle^s$ ABY, AQB are rt. $\angle^s$,

also the $\angle$APQ=$\angle$ABQ [*Th* 39]=comp^t of $\angle$BAQ [*Th* 16]

=$\angle$AYB, [*Th* 16]

and $\angle$PAQ is common to the $\triangle^s$ PAQ, YAX

the $\triangle^s$ are equiangular, and similar

(ii) $\angle$QYX+$\angle$QPX=$\angle$APQ+$\angle$QPX=2 rt $\angle^s$

P, Q, Y, X are concyclic. [*Th* 40, *Converse*]

7 Let CA be produced to K, and let the bisector of $\angle$KAB meet the base in D and the $\bigcirc^{ce}$ in E [In the fig considered E, D lie on opposite sides of A] Join BE

Then $\angle$BAE=$\angle$EAK [*Hyp*]

=$\angle$CAD [*Th* 3]

Also $\angle$AEB=$\angle$ACD, [Ex 5, p 163]

$\triangle^s$AEB, ACD are similar,

AB AD=AE AC

rect AB, AC=rect AE, AD [*Th* III, p 250]

8 See Definition, p 277

Here AB XB=AX^2 Let AX=x cm, and XB=$(10-x)$ cm

$10(10-x)=x^2$, or $x^2+10x-100=0$

From this $x=\pm5\sqrt{5}-5$

=6 18 , rejecting the negative value for *internal* section

greater segment=6·2 cm, to nearest mm

9 Let ABC be given $\triangle$ Along AB, AC mark off AD, AE respectively, making AD=AE=the *mean* proportional between AB, AC [*Prob* 38] Join DE

Then AB AD=AE AC, [*Constr*]

AB AC=AD AE

But $\triangle$ABC $\triangle$ADE=AB AC AD AE, [*Th* 71]

$\triangle$ABC=$\triangle$ADE

And since AD=AE the $\triangle$ADE is isosceles

10. Let ABC be the given $\triangle$. Along BC measure off BD = given base.

At B draw BQ perp to BC and equal to the altitude of the given $\triangle$. Join DQ. Through C draw CP par[1] to DQ and meeting BQ, or BQ produced, in P. Join PD, QC.

Then BP : BQ = BC : BD, and $\therefore$ BP : BD = BQ : BC,

$\therefore$ $\triangle$ PBD = $\triangle$ QBC [*Th.* 71] = $\triangle$ ABC.

Through P draw PR par[1] to BD to meet the perp. bisector of BD in R. Then $\triangle$ RBD is isosceles [*Prob.* 14] and equal to $\triangle$ PBD [*Th.* 26], that is, equal to $\triangle$ ABC.

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1. See Ex. 2, p. 295.

2. The $\triangle^s$ APQ, AXY, ABC are all similar. [*Th.* 14 and 62.]

$\therefore$ $\triangle$ APQ : $\triangle$ AXY : $\triangle$ ABC = AP² : AX² : AB². [*Th.* 72.]

But $\triangle$ APQ = $\frac{1}{3}$ $\triangle$ ABC, and $\triangle$ AXY = $\frac{2}{3}$ $\triangle$ ABC,

$\therefore$ $\triangle$ APQ : $\triangle$ AXY : $\triangle$ ABC = 1 : 2 : 3,

$\therefore$ AP : AX : AB = 1 : $\sqrt{2}$: $\sqrt{3}$.

The general construction may be illustrated by dividing the triangle into *three* equal parts.

Divide AB into *three* equal parts at p and x, and on AB draw a semi-circle.

At p and x draw perp[s] to AB to cut the O[ce] at p' and x'.

From centre A with radii Ap', Ax' cut AB at P and X.

Through P and X draw PQ, XY par[1] to BC.

Then as in Note, p. 276,

AP is a *mean* proportional between Ap and AB,

AX is a *mean* proportional between Ax and AB.

So that AP² = Ap . AB = $\frac{1}{3}$ AB², $\therefore$ AP = $\frac{1}{\sqrt{3}}$. AB,

and AX² = Ax . AB = $\frac{2}{3}$. AB², $\therefore$ AX = $\frac{\sqrt{2}}{\sqrt{3}}$ AB,

or, AP : AX : AB = 1 : $\sqrt{2}$: $\sqrt{3}$.

By dividing AB into n equal parts and proceeding as above, parallels may be drawn dividing the $\triangle$ into n equal areas.

3. Apply Prob 40

4 The $\triangle^s$ BAD, BCD are congruent (*Th* 7),

∴ area of given quad$^l = 2\triangle$ BAD $=$ AB AD $= 48$ sq cm

new quad$^l = \frac{36}{48}$, or $\frac{3}{4}$, of the original quad^l

Now apply Prob 40

5 Suppose the given ○ divided as required, and let any radius OA cut the two inner circles in B and C respectively

Now the areas of the three ○^{s} are in ratio 1 2 3

$OC^2 \; OB^2 \; OA^2 = 1 \; 2 \; 3$ (for the areas of ○^{s} are proportional to the squares of their radii)

Construction Trisect OA, and set off OP $= \frac{2}{3}$OA, and OQ $= \frac{1}{3}$OA

On OA draw a semi-circle, and, as in Note, p 276, find OB the mean proportional between OA and OP, and OC the mean proportional between OA and OQ Thus the positions of B and C are determined

The proof follows as in Ex 2

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1 The $\angle^s$ OPA, AOQ are both rt $\angle^s$

$$AP \; AQ = AO^2 \text{ [Th 66, Cor]} = \text{a constant}$$

Here AO $= 3\,4''$ ∴ AP AQ $= 11\,56$ sq in

2 The $\angle$TQP is a rt $\angle$ [*Ex* 7, p 177]

The $\angle$CTP is a rt $\angle$ [*Th* 46]

(i) $CQ \; CP = CT^2 = 100$ [*Th* 66, *Cor*]

(ii) $PQ \; PC = PT^2$

$= CP^2 - CT^2$

$= (24^2 + 10^2) - 10^2$ [*Ex* 1, p 133]

$= 576$

(iii) $TQ \; CP = 2\triangle PTC = CT \; TP$

$$TT' = 2TQ = 2 \times \frac{10 \times 24}{26} = \frac{240}{13} = 18\,46$$

for $PT = 24$ from (ii), and $CP^2 = 24^2 + 10^2 = 26^2$

3 Let A, B be the two centres, P and Q the common pts
Let PQ cut AB in X, then PX=XQ and ∠PXA=90°
[III p 143]

(i) Denote the length of AX by a BX=1 7 − a

Also $AP^2 - AX^2 = PX^2 = BP^2 - BX^2$ [*Th* 29]

$(2\ 6)^2 - a^2 = (2\ 5)^2 - (1\ 7 - a)^2$ whence $a = 1\ 0$

OX=OA+AX=1 3

And $PX^2 = (2\ 6)^2 - (1\ 0)^2 = 5\ 76$, PX=2 4

P is the pt (1 3, 2 4),

Q, the image of P in OX, is (1 3, −2 4)

(ii) PQ=2PX=4 8

(iii) Let T be the pt (1 3, 3 4), TM, TN the tangents from T to the two circles Since T lies on QP produced

$TM^2 = TP\ \ TQ = TN^2$ [*Th* 58]

Also TP=TX − PX =3 4 − 2 4=1 0,

TQ=TX+XQ=3 4+2 4=5 8

TM=TN=$\sqrt{5\ 8}$=2 41

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1 Draw AD perp to base BC

Then BA AX=rect contained by AD and diam of ○ BAX [*Th* 77]

And CA AX=rect contained by AD and diam of ○ CAX

But BA=CA

diam of ○ BAX=diam of ○ CAX

2. The △ˢABD, ACD are identically equal [*Th* 18] BD=CD

Also A, B, D, C are concyclic [Converse of *Th* 40]

BC AD=AB CD+AC BD [*Th* 78]

=twice AB BD

3. Let diagonals AC, BD intersect at rt $\angle^s$ in E

Then sum of rect^s of opp sides $= AC\ BD$ [*Th* 78]

$$= AC(DE + EB) = AC\ DE + AC\ EB$$
$$= 2\triangle ADC + 2\triangle ABC$$
$$= \text{twice area of } ABCD$$

4 Let BD bisect AC in E Draw AX, CY perp to BD

Then AB AD = rect contained by AX and diam of ○ [*Th* 77]

Also BC CD = rect contained by CY and diam of ○

But AX = CY, from the congruent $\triangle^s$ EAX, ECY,

$$AB\ AD = BC\ CD$$

5 Draw AD perp to BC and let X be any pt in BC

Then AB AX = rect contained by AD and diam of ○ ABX

Hence AD AX = AB diam of ○ ABX [*Ex* 12, p 253]

Similarly AD AX = AC diam of ○ ACX

AB diam of ○ ABX = AC diam of ○ ACX,

or, AB AC = diam of ○ ABX diam of ○ ACX

6 Let BC be the given base On BC describe a segment of a ○ containing an $\angle$ equal to given $\angle$ Let X, Y be the sides of given rectangle To the diameter, X and Y, find a fourth proportional DA Place DA in segment perp to BC Then BAC is the required $\triangle$ [*Th* 77]

7 Let ABC, DEF be the two equal $\triangle^s$, and let AM, DN be perp^s from the vertices A, D upon the bases BC, EF Let PQ be the diameter of the ○ circumscribing the $\triangle^s$ABC, DEF

Then rect BA, AC = PQ AM

And rect ED, DF = PQ DN

rect BA, AC rect ED, DF = AM DN

But $\triangle BAC = \frac{1}{2}BC\ AM$, and $\triangle DEF = \frac{1}{2}EF\ DN$

BC AM = EF DN, AM DN = EF BC [*Ex* 12, p 253]

rect BA, AC rect ED, DF = EF BC

8 Here $PB\ CA+PC\ AB=PA\ BC$ [*Th* 78]

But $BC=CA=AB$ $\quad (PB+PC)\ AB=PA\ AB$

$$PB+PC=PA.$$

9 Because $\angle ABD=\angle CBD$, arc AD=arc CD,

chord AD=chord DC And because A, C are fixed,

D is a fixed point, and AD is constant

But $AB\ CD+BC\ AD=AC \cdot BD$, [*Th* 78]

or, $(AB+BC)\ AD=AC\ BD$ [*Th* 50]

$AB+BC\ BD=AC\ AD$=constant [*Ex* 12, p 253]

10 To find Δ use either the method of p 111 or the formula proved in Ex 7 (iii) of that page

Page 306.

1 (i) The $\angle ACD=\frac{1}{2}\angle ACB$ [*Th* 7]

$=\angle APB$ [*Th* 38]

Similarly $\angle ADC=\angle AQB$

$\angle CAD=\angle PBQ$ [*Th.* 16]

(ii) Similarly $\angle CBD=\angle PBQ$

Hence $\angle CBP=\angle DBQ$

$\angle BPC=\angle BQD$

2 Let O be the centre of the ○, and Y the middle point of CD Join XY, OC, OY.

Then OY is perp to AB [*Th* 31 and *Hyp*]

Now $XC^2+XD^2=2\{CY^2+XY^2\}$ [*Th* 56]

$=2\{CY^2+OY^2+OX^2\}$ [*Th* 29]

$=2\{OC^2+OX^2\}$

$=2\{OA^2+OX^2\}$

$=XA^2+XB^2$ [*Ex* 6, p 230]

3 Let ABCD be the quad[l] Let AB, DC meet at P, and BC, AD at Q. Let the bisectors of the $\angle^s$ at P and Q meet at O Join PQ.

Then $\angle OPQ = \frac{1}{2}(\angle CPQ + \angle APQ)$,

and $\angle OQP = \frac{1}{2}(\angle CQP + \angle AQP)$,

$\angle POQ = \frac{1}{2}(\angle PCQ + \angle PAQ)$ [*Th* 16]

$= \frac{1}{2}(\angle BCD + \angle BAD)$ [*Th* 3]

$=$ one rt angle [*Th* 40]

4 Let PAB be the given vertical angle, AB the given side, and K the given altitude

From centre A, with radius K, describe a ○, and from B draw BDC to touch the ○ at D, and meet AP at C

Then ABC is the required triangle

For AD is perp to BC [*Th* 46], and is equal to K

5 On AC describe a semicircle AQC At B make $\angle ABQ = 45^\circ$ Draw QP perp to AC Then PB=PQ. But PQ is a mean proportional between PA and PC [*Prob* 38] P is the required point

6 By par[ls], BF FA=BD DC=AE EC

And $\triangle$BFD $\triangle$AFE=BF FA [*Th* 70]

And $\triangle$AFE $\triangle$CDE=AE EC

$\triangle$BFD $\triangle$AFE=$\triangle$AFE $\triangle$CDE

7 Join PY, QX [In the fig taken PX and QY are on opposite sides of PQ.]

Then $\angle QPX = \angle PQY$ [*Th* 14]$= \angle PXY$ [*Th* 39]

Also $\angle QPY = \angle QXY$ [*Th* 39]

By addition, $\angle XPY = \angle PXQ$

But $\angle PXQ$ is constant, since PQ is fixed

$\angle XPY$ is constant, arc XY is constant [*Th* 42]

chord XY is of constant length [*Th* 45]

XY touches a fixed concentric circle [*Th* 34]

8 Join AQ, and produce it to meet A′P′ at X Join A′Q′.

Then, by hyp and Theor 41, the figs XP, XQ′ are rects

$$AX = P'P, \text{ and } A'X = Q'Q.$$

Now $AA'^2 = AX^2 + A'X^2$ [*Th* 29]

$$= P'P^2 + Q'Q^2$$

9 The diameters of the ⊙s about ABE, ACE are in the ratio of AB to AC [*Ex* 5, p 305] But, because AE bisects ∠BAC,

$AB : AC = BE : EC.$ [*Th* 61]

10 Because DE is par[1] to the tangent at A, it makes with AB, AC angles respectively equal to ACB, ABC [*Th* 49],

or ∠ADE = ∠ACB and ∠AED = ∠ABC.

△s ABC, AED are equiangular,

$$AB : AE = AC : AD;$$

rect AB, AD = rect AC, AE.

11 Let C, D be two given pts on AB On AD, BC as diameters describe semicircles cutting in P Draw PX perp to AB

Then $AX \cdot XD = PX^2$ [*Th* 66, *Cor*] $= CX \cdot XB$

$\dfrac{XA}{XB} = \dfrac{XC}{XD}$ [*Ex* 12, p 253]

X is the required pt

12 Let D, E, F be the feet of the perps

Bisect ∠s DEF, EFD, FDE, by lines meeting at O [II p 96]

Draw lines through D, E, F perp to OD, OE, OF.

Then any one of the four △s ABC, OBC, OCA, OAB thus formed will satisfy the given conditions [See II, p 208]

13 Let ABCD be the quadrilateral whose sides AB, BC, CD, DA touch the inscribed ⊙ at X, Y, Z, W

Let AB, DC, produced, meet at P, and AD, BC, produced, meet at Q Bisect the ∠s at P and Q by PO, QO Join PQ.

Then PO is perp to XZ, and QO to YW [*Ex* 7, p 177]

Now, as in Ex 3, p 306, it may be shewn that

$$\angle POQ = \tfrac{1}{2}(\angle BCD + \angle BAD)$$

$$= \text{one rt angle} \qquad [\textit{Th } 40]$$

PO and QO are at right angles to each other

XZ and YW are at right angles to each other

14 Let OA, OB be the two given st lines, C and D the centres of the given ○ˢ, and P their point of contact Then P is the middle point of CD [*Hyp* and *Th* 48] Also it is clear that C and D will move on st lines parˡ respectively to OA, OB, and at a distance from them equal to the radius of the given ○ˢ

If these lines intersect at X, the locus of P is a ○ whose centre is X, and whose radius is equal to the radius of either of the given ○ˢ For XP=PC=PD [*Th* 41]

15 For let O be the centre of the given ○ Join OC

Then $\qquad \angle EDC = \angle BAC = \angle OCE \qquad$ [*Th* 39]

Hence OC is a tangent to the ○ DEC [*Th* 49 *Converse*]

And since OC is a radius of the given ○, the two ○ˢ cut orthogonally

16. Let ABCD be the quadrilateral Let the bisectors of the ∠ˢ A, B meet at X, of the ∠ˢ B, C at Y, of the ∠ˢ C, D at Z, and of the ∠ˢ D, A at W

Then $\qquad \angle AXB = 180° - (\tfrac{1}{2}A + \tfrac{1}{2}B) \qquad$ [*Th* 16]

Also $\qquad \angle CZD = 180° - (\tfrac{1}{2}C + \tfrac{1}{2}D)$

$$\angle AXB + \angle CZD = 360° - \tfrac{1}{2}(A+B+C+D) = 180°$$

$$\angle YXW + \angle YZW = \text{two rt angles},$$

the points X, W, Z, Y are concyclic.

17 Because $\qquad AB \quad AC = AC \quad AD,$

$$AB \quad AB - AC = AC \quad AC - AD,$$

that is, $\qquad AB \quad BC = AC \quad CD,$

$$AB \quad AC = BC \quad CD$$

Again $\qquad AB \quad AE = AE \quad AD \qquad$ [*Hyp*]

in △ˢ ABE, AED, the sides about the common ∠ at A are proportional these △ˢ are similar [*Th* 64]

$$BE \quad ED = AB \quad AE = AB \quad AC = BC \quad CD$$

CE bisects ∠ BED

18 With the given diameter EB describe a ○ EABC Make ∠BEC = given vertical ∠ Divide BC in given ratio at D Bisect arc BC in F Produce FD to A ABC is the required △ For ∠s BAC, BEC in same segment are equal, and since

arc BF = arc CF, ∠BAF = ∠CAF

BA AC = BD DC = given ratio

19 Through Q draw a st line par[1] to the given st line This is the required locus [Proof by Theor 60]

20 Let C be the centre of the given ○ In OC take D, so that OD OC = given ratio Then △s OPC, OQD are similar, and DQ CP = given ratio But CP is constant, and D is fixed, locus of Q is a ○, having centre D and radius DQ.

21. Let CD be the perp and let CD meet the first ○ at G, and the second ○ at O and O′

Since the ○s are equal, the distances from D of the two points on the one ○ are respectively equal to the distances of the two points on the other

Let O be the point corresponding to G

Then O is the orthocentre [*Ex* 1, p 209], for DO = DG

If C is within the second ○, then O′ is the orthocentre

Otherwise The ∠AGB is the supplement of the ∠ACB [*Th* 40]

And since the segments AGB, AOB are equal [*Hyp* and *Th* 44]

∠AGB = ∠AOB, ∠AOB is supp[t] of ∠ACB

orthocentre is on arc AOB [III p 210] But the orthocentre is on perp CD orthocentre is at O

22 Call the ○s (i), (ii), (iii) Let (i) and (iii) intersect again at B, (ii) and (iii) at C, (i) and (ii) at D

Then as in the second proof of the last exercise it may be shewn by means of the ○s (i) and (iii) that the orthocentre of the △ABC lies on the arc ADB Similarly, by means

of the ○^{s} (ii) and (iii) the orthocentre of the $\triangle$ABC lies on the arc ADC

the orthocentre is at D

Hence of the four points A, B, C, D each is the orthocentre of the triangle formed by joining the other three [*Ex* 4, p 209]

23 Let A be the given point, and O the centre of the given ○ Join AO, and bisect it at X With centre X and radius equal to one-half of the radius of the given ○, describe a ○ cutting the convex ○^{ce} at Y

Join AY and produce it to meet the concave ○^{ce} again at B

Then AY=YB, for X is the middle point of the side AO, and XY is half the base OB [See *Ex* 2 and 3, p 64, and *Th* 65]

Impossible when the minimum distance from A to the ○^{ce} is greater than the diameter

24 Let AB be the given base, H the given altitude, and K the radius of the circum ○

Draw PQ par^l to AB and at a distance from it equal to H Then the vertex of required $\triangle$ lies on PQ

Let ○^{s} with centres A, B and radii equal to K intersect at S, on the same side of AB as PQ

From centre S, with radius K, intersect PQ at C or C′ Then either of the $\triangle^s$ ABC, ABC′ satisfies the conditions

25 Let ABC be a triangle of the system on the fixed base AB.

Produce AC to D, making CD equal to CB

Then AD is of constant magnitude Join BD cutting the bisector of the $\angle$BCD at P Then CP bisects BD at rt angles Required the locus of P

Bisect AB at O Join OP Then OP=$\frac{1}{2}$AD [*Ex* 3, p 64]

That is, OP is constant, and since O is a fixed point, the locus of P is a circle, whose centre is at O, and whose radius is half AD

26 It has been proved in VII p 214, that if I, I_1, I_2, I_3 are the centres of the inscribed and escribed circles of the $\triangle$ABC, each of these four points is the orthocentre of the triangle formed by the other three, and that the original $\triangle$ABC is the pedal triangle

Hence given any three of the points I, I_1, I_2, I_3, we have only to draw the pedal triangle of the triangle so formed

27 Draw CD perp to BC to meet the circum-⊙ at D. Join AD. BD.

Then BD is a diam., and AD is perp to BA. [*Th.* 41.]

But CO produced is also perp to AB [*Hyp.*]; ∴ AD and CO are par^l. And by construction AO and DC are par^l; ∴ AO = DC. [*Th.* 21.]

But, since BCD is a rt. ∠, $BD^2 = BC^2 - CD^2$. [*Th.* 29.]

$$\therefore d^2 = BC^2 - AO^2$$

28. Because C bisects arc AB, chord AC = chord BC.

But AD . BC − DB . AC = AB . DC [*Th.* 78.]

AC . (AD − DB) = AB . DC

∴ AD − DB : DC = AB : AC [*Ex.* 12, p. 253]

29. Let BD, CE cut in O. Because BO : OD = CO : OE,

∴ DE is par^l to BC, and △^s BOC, DOE are similar:

BC : DE = BO : OD = 4 : 1

But BA : EA = BC : ED = 4 : 1

∴ BA − EA : EA = 4 − 1 : 1, [IV. p. 250.]

or BE : EA = 3 : 1

30 Let P, Q be two fixed p^ts, AB any st. line between them. Draw PM, QN perp^s on AB, and let PQ cut MN in O.

Then OP : OQ = PM : QN = constant

∴ MN always passes through the fixed pt. O, which divides PQ internally in the given constant ratio.

31. Make ∠CAD = ∠ABC. Then △^s BDA, ADC are similar.

∴ BD : DA = DA : DC

DA is a mean proportional between BD and DC.

32. The common tangent at O makes with OA an ∠ equal to ∠OQP and to ∠OBA [*Th.* 49]. ∴ ∠OQP = ∠OBA. ∴ PQ is par^l to AB. ∴ ∠PQC = alt. ∠QCB = ∠CPQ in alternate segment. ∴ chord CQ = chord CP. ∴ OC bisects ∠BOA. [*Th.* 44, 43.]

∴ OP : OQ = OA : OB = AC : BC [*Th.* 61.]

33 Taking the Figure in which D is within the $\odot$ and on the side of O remote from AB, join OA, OC Then the $\angle CEO = 90° - \angle B = 90° - \frac{1}{2}\angle COA$ at centre $= \angle OCD$

$\triangle^s$ ODC, OCE are equiangular

$OD : OC = OC : OE$

$OD \cdot OE = OC^2$

34. Join AD, BD Then the $\angle BDY = \angle BAD$ [*Th* 49] $= \angle BDX$

DB bisects $\angle YDX$ internally Again, DA is perp to DB,

DA bisects $\angle YDX$ externally.

$XB : BY = XD : DY = XA : AY$ [*Th.* 61]

$BX : AX = BY : AY$

35 Let P, Q be the given pts. Divide PQ, internally and externally, at A and B in the given ratio [*Prob* 37] On AB as diameter describe a $\odot$ Then the distances of P and Q from any pt on this circle are in the given ratio [*Ex* 8 p 259] The pt or pts, if any, where this $\odot$ cuts the given $\odot$ are the pts required.

36 Let the bisector of the vert $\angle$ be a part of PQ, a line of unlimited length. Bisect AB, the given base, at rt angles, by a line which cuts PQ at X

Describe a $\odot$ about AXB, and let it cut PQ again at C Then the $\triangle ABC$ will be that required.

For chord AX = chord BX [*Th.* 4]

. arc AX = arc BX [*Th* 44]

. $\angle ACX = \angle BCX$ [*Th* 43]

37 Let O, O′ be the centres Then, because BE and C′O′ are both at rt. $\angle^s$ to ABC′,

$AB : BC' = AE : EO'$ [*Th* 60]

$AB : 2BC' = AE : EA'$

Similarly $A'B : 2B'C = A'E : EA$

$AB : 2BC = 2B'C : AB$

$AB \cdot A'B = 4\, BC' \cdot B'C$

38. Let P be the given pt and O the centre of the given ⊙
On OP describe a segment of a ⊙ containing an angle of 45° [*Prob* 24] Let this cut the given ⊙ in Q Join PQ cutting the ⊙^{ce} again in R

Then $\angle ORQ = \angle OQR = 45°$

$\angle QOR = 90°$ [*Th* 16]

39. With figure of p 207, let S be the centre of circum ⊙, and let SA meet EF at X

Then $\angle AFX = \angle ACB$ [II *Cor* ii, p 208]

And $\angle ASB =$ twice $\angle ACB$, [*Th* 38]

$\angle SAB = 90° - \angle ACB$ [*Th* 5, 16]

Hence from $\triangle$ AFX, the $\angle$AXF is a rt angle [*Th* 16]

40 Since the $\angle$s CEP, CDP are rt angles, the ⊙ about the $\triangle$ PED passes through C, and is described on PC as diam Hence it is required to find the locus of X, the middle point of CP

Take S the centre of the ⊙ and join SX

Then since the $\angle$CXS is a rt angle [*Th* 31], and the points C, S are fixed, the locus of X is a ⊙ on CS as diam

41 Take the figure of p 212 Draw the diam AX, and join AP, PX Then the four points P, D, E, C are concyclic

$\angle EDB = \angle EPC$ [*Ex.* 5, p 163]

$= 90° - \angle PCE$

$= 90° - \angle PXA$ [*Th* 39]

$= \angle PAX$ [*Th* 41]

42. On the given base AB describe a segment containing the given angle, and another segment containing half the given angle Make the $\angle$ABK half the given diff of the base angles and draw BD perp to BK to cut the larger segment at D Join AD, cutting the smaller segment at C and BK at Q Then ABC is the $\triangle$ required

For $\angle ACB = \angle CDB + \angle CBD$, and $\angle CDB = \frac{1}{2}\angle ACB$,

$\angle CBD = \frac{1}{2}\angle ACB$ $\angle CDB = \angle CBD$, $CD = CB$

And KBD is a rt angle Hence CQ = CB [*Th* 41]

BK is perp to the bisector of the vertical $\angle$ACB

$\angle ABK = \frac{1}{2}$ diff of $\angle$s CBA, CAB [*Ex* 2, p 138]

43 Let ACD AEF and BEC, BFD be the two pairs of lines Let the ⊙s about the △s ACE, BEF meet at P Then shall the ⊙s about the △s AFD BCD pass through P Join PF, PE, PA

Then $\angle$BFP = $\angle$BEP [*Th* 39]

= $\angle$PAC [*Ex* 5, p 163]

Hence $\angle$s PAD, PFD together = two rt. angles

∴ the points A, D, F P are concyclic that is the ⊙ about the △ ADF passes through P

The proposition may also be proved by the properties of Simson's Line [See V and *Ex* 3, p 212]

44. From the last exercise it is seen that the ⊙s about the four △s pass through a common point P Hence it may be seen (*Ex* 3 p 212) that the four △s have a common pedal for the point P

Also (*Ex* 4 p 212) this pedal bisects each of the lines joining P to the four orthocentres

Hence, by the method of Ex. 9, p 94, the orthocentres are collinear

45 For suppose any two consecutive sides AB BC of an inscribed polygon are unequal Let P be the middle point of the arc AC

Then AP, PC are together greater than AB BC [See *Ex* 19, p 318], and △ APC is greater than △ ABC for it has a greater altitude

Hence there is an inscribed polygon which has a greater perimeter and a greater area than the given polygon

∴ an inscribed polygon cannot have the maximum perimeter or maximum area unless every pair of consecutive sides are equal, that is, unless it is regular

46 Let E be the middle pt of AB Then, since diagonals of a par^{m} bisect one another E is middle pt of CD

Draw DO, par^{l} to EP, meeting CP in O Then △s DOC, EPC are similar ∴ OC = twice PC, and OD = twice PE, hence O is a fixed point, and OD is of constant length ∴ the locus of D is a ⊙, having the fixed pt O as centre

47 Let ABC be the isosceles △ Draw AD perp to the base BC At A make the ∠s DAE, DAF each 30°, E and F being in BC Then AEF is an equilateral △ From AE, AF cut off AG, AH each equal to the mean proportional between AE and BC Then by similar △s AGH, AEF,

$$\triangle AEF \quad \triangle AGH = AE^2 \quad AG^2 = AE^2 \quad AE \quad BC \qquad [Th\ 72]$$
$$= AE \quad BC$$
$$= EF \quad BC$$
$$= \triangle AEF \quad ABC$$

the equilateral △ AGH = given △ ABC

48. Let MAN be the given vert ∠ Along AM, AN take AP, AQ each equal to half the given sum of the sides containing the vert ∠

Let ABC be one △ of the system Then clearly PB = CQ

At P and Q draw PX, QX perp to AM and AN Then X is a fixed point, and it may be shown [*Th* 18] that PX = QX Hence the △s BPX, CQX are identically equal [*Th* 4]

the ∠PXB = the ∠QXC to each add the ∠PXC;

then the ∠BXC = the ∠PXQ.

But since the ∠s at P and Q are rt angles, the ∠s PXQ, PAQ are supplementary, the ∠s BXC, BAC are supplementary

X is on the ○ about the △ABC Thus the ○ passes through two fixed points A and X Hence the locus of the centre is the st line bisecting AX at rt angles

49 Let D, E, F be the vertices of the equilateral △s Then the △s BAE, FAC are identically equal BE = FC But the △s ZAB, YAC are similar,

$$AZ \quad AY = AB \quad AC$$
$$AZ \quad AF = AY \quad AC$$

But ∠ZAY = ∠FAC △s ZAY, FAC are similar

$$ZY \quad FC = AY \quad AC$$

Similarly $\qquad XY \quad BE = AY \quad AC$

Hence $\qquad XY = YZ = ZX$

50 Let XY be the given st line, and P, Q the given pts [In the fig considered P is outside the given ○] Join PQ and

in it take a pt F so that rect PF, PQ = the rectangle contained by the segments of any chord of the circle through P [*Prob* 35] Let QP and YX be produced to meet at Z Let K be the length of a chord of the ○ which subtends at the ○ce an angle equal to the ∠QZY, through F draw a line FBD cutting off a chord BD equal to K [by using Theor 34 and Ex 2, p 177] Draw PBA meeting ○ in B, A, and join AQ meeting the ○ in C Then ABC is the required △

Because rect PF, PQ = rect PB, PA,

PF PA = PB PQ, [*Ex* 12, p 253]

△s PFB, PAQ are similar [*Th* 64]

∠PFB = ∠PAC

= ∠BDC (or the supplement of BDC,)

DC is par[l] to PQ.

And because ∠DCB = ∠QZY,

BC is par[l] to XY

51 Let P, Q, R be the given pts Join PQ and determine a pt F in it as in Ex 50

In the circle inscribe a △DBC so that DB and BC pass through F and R respectively, while DC is par[l] to PQ [*Ex* 50]

Produce PB to meet the ○ce in A, join QA meeting the ○ce in C′, and join DC′

Then ∠BAQ = ∠FDC′ in the same segment

Also, as in Ex 50, the △s PFB, PAQ are similar,

∠PAQ = ∠PFB = alt ∠FDC

∠FDC = ∠FDC′

Hence C′ coincides with C, and the △ABC fulfils the required conditions

52 Take the case in which the points are in the following order O, A, B, X, Y

Take OE a mean prop[l] between OA and OY and describe a ○ with O as centre and OE as radius Take P on the ○ce of this ○, describe a ○ round PAY and also round PBX Then OP touches each of these ○s, since

$OP^2 = OA \cdot OY = OB \cdot OX$

∠OPB = ∠PXB [*Th.* 49]

But $\angle$OPB = sum of $\angle^s$ OPA, APB,

and $\angle$PXB = sum of $\angle^s$ XPY, PYA [*Th* 16]

sum of $\angle^s$ OPA, APB = sum of $\angle^s$ XPY, PYA,

but $\angle$OPA = $\angle$PYA [*Th* 49]

$\angle$APB = $\angle$XPY

53 Let OA, OB be the two given lines Produce AO to A′, and in OA′ and OB take points H and K, so that OK OH = the given ratio

Draw OC par^l to HK OC is the required locus

For, draw KQ perp to OB to meet OC in Q, and QR perp to OA Also, from any pt P in OC, draw PM perp to OA and PN perp to OB

Then PM PN = QR QK

But $\triangle^s$ OHQ, OKQ on same base OQ and between same par^{ls} are equal

rect QR, OH = rect QK, OK

QR QK = OK OH

PM PN = given ratio

54 Let ABC be the triangle, S, I the centres, and R, r the radii of the circumscribed and inscribed circles

(i) To prove $SI^2 = R^2 - 2Rr$

Join AI, and produce it to meet the O^{ce} of the circum-O at X Join XS, and produce it to meet the O^{ce} again at Y Join XC, and draw IE perp to AC Join YC

Then in the $\triangle^s$ IAE, XYC,

$\angle$IAE = $\angle$XYC [*Th* 39], and $\angle$IEA = $\angle$XCY [*Th* 41],

hence the $\triangle^s$ IAE, XYC are equiangular, [*Th* 16]

IE XC = IA XY, [*Th* 62]

IE XY = XC IA

But IE = r, XY = 2R, and XC = XI, [*Ex* 16, p 206]

$2Rr$ = XI IA.

Join SI, and produce it both ways to meet the O^{ce} at P, Q.

Hence XI IA = PI IQ [*Th* 57]

= (PS + SI)(SQ − SI)

= $R^2 - SI^2$,

hence $SI^2 = R^2 - 2Rr$

(ii) Similarly, if I_1, I_2, I_3 are the centres and r_1, r_2, r_3 the radii of the escribed ⊙s, it may be shewn that

$$SI_1^2 = R^2 + 2Rr_1, \quad SI_2^2 = R^2 + 2Rr_2,$$

and $$SI_3^2 = R^2 + 2Rr_3$$

(iii) To prove $IN = \frac{R}{2} - r$ (Feuerbach's Theorem)

We here give an outline of Feuerbach's proof, one step of which depends on trigonometrical work

Let S, I, and N be the centres of the circumscribed, inscribed, and nine-point ⊙s of the △ ABC, and O its orthocentre Let AO meet BC at D, and the ⊙ce of the circumscribed ⊙ at G Join SI, IO, and SO, and let SO produced both ways meet the ⊙ce at P and Q.

Then N is the middle point of SO [p 211]

And since IN is a median of the △ SIO,

$$SI^2 + IO^2 = 2IN^2 + 2SN^2,$$

or $$SI^2 + IO^2 = 2IN^2 + \tfrac{1}{2}SO^2 \qquad (1)$$

But $SI^2 = R^2 - 2Rr$, and $SO^2 = R^2 - PO \cdot OQ$ [*Th* 53]

$$= R^2 - AO \cdot OG$$

Also it may be proved by trigonometry from the △ IAO that

$$IO^2 = 2r^2 - AO \cdot OD$$

$$= 2r^2 - \tfrac{1}{2}AO \cdot OG \qquad [\textit{Ex}\ 1,\ p\ 209]$$

Substituting these results in (1), we have

$$2(R^2 - 2Rr) + 4r^2 - AO \cdot OG = 4IN^2 + R^2 - AO \cdot OG,$$

or, $$R^2 - 4Rr + 4r^2 = 4IN^2,$$

i e, $$(R - 2r)^2 = (2IN)^2,$$

$$\frac{R}{2} - r = IN$$

Remembering that the radius of the nine-points ⊙ is half the radius of the circumcircle we see that the nine-points ⊙ touches the inscribed ⊙

(iv) Similarly we may prove that $NI_1 = \frac{R}{2} + r_1$, and similar formulæ for NI_2, NI_3, whence the nine-pts ⊙ touches the 3 escribed ⊙s

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1. Let the given circle cut the axis of y in B, B′. Through the given pt A (20, 0) draw FAF′ par[l] to OY. Join AB, AB′ cutting the ⊙[ce] in E, E′ respectively. Produce OE, OE′ to meet AF in F and F′.

Then the ⊙[s] whose centres are F, F′ and radii FA, F′A respectively will each satisfy the req[d] conditions.

For ∠FAE = ∠OBE [*Th* 14] = ∠OEB [*Th* 5] = ∠AEF,

$$FE = FA$$

Let $AF = x$, then $OF = OE + EF = 10 + x$

But $OF^2 = OA^2 + AF^2$, $\quad (10+x)^2 = 20^2 + x^2$, whence $x = 15$

radius of the ⊙ = 15

From E draw ED perp to OX

Then $OD : OA = ED : FA = OE : OF$ [*Th* 62]

But $OF = OE + EF = 25$, $\quad OE : OF = 2 : 5$,

hence $OD = 8$, $ED = 6$

coords of E are (8, 6)

2. Let A be the given pt (6, 8). Draw AP the tangent at A cutting OY in P. With centre P, and radius PA, describe a ⊙ cutting OY on K, K′ (K being outside the ⊙). From K, K′ draw perp[s] to OY meeting OA in C, C′. Then the ⊙[s] whose centres are C and C′, and radii CK, C′K′ respectively, satisfy the required conditions.

For the rt-∠[d] Δ[s] CKP, CAP are congruent [*Th* 18], and CK = CA

Now if AM be the perp from A on OY, we have, by Theor 62,

$$OC : CK = OA : AM = 10 : 6$$

, *dividendo*, $OC : OC - CK = 10 : 10 - 6 = 5 : 2$

Hence, remembering that $OC - CK = OC - CA = OA = 10$, we get $OC = 25$, and $CA = 15$

$$OK = \sqrt{OC^2 - CK^2} = \sqrt{25^2 - 15^2} = 20$$

radius of the larger ⊙ is 15, and its pt of contact is (0, 20)

Similarly smaller ⊙ is $3\frac{3}{4}$, (0, 5)

3 *Analysis* Let BOC be the given quadrant, A being the mid-pt of the arc BC Then by symmetry the req^d ○ will touch the arc at A Let M be its pt of contact with OC Draw AT touching the quadrant at A and cutting OC produced at T Then TM = TA.

Construction Draw AT touching the arc at its mid-pt A and cutting one of the bounding radii OC at T With centre T and radius TA cut this radius at M From M draw MP perp to OC and meeting OA in P Join PT Then the $\triangle^s$ TMP, TAP are congruent [*Th* 18] Hence PM = PA, and by Theor 18, PM = perp from P on OB
a ○ whose centre is P and radius PM is the req^d ○

Let radius $= r$ inches, then OM = PM $= r$, and OP $= 2 - r$

But $OP^2 = OM^2 + MP^2$, $(2-r)^2 = r^2 + r^2$, whence $r^2 + 4r - 4 = 0$ Solving the equation we get $r = -2 \pm 2\sqrt{2}$, whence, rejecting the negative root, the radius $= (2\sqrt{2} - 2)$ ins $= 0\,83''$, nearly

4 Let A be the pt (2, 2) Join OA, and draw AT perp to OA to meet OY in T From TO and TY cut off TP and TQ respectively, each equal to TA Let the perps from P and Q to OY meet OA, or OA produced, in C and C′

Then, from the congruent $\triangle^s$ TPC, TAC, we have CP = CA, also C′Q = C′A

Hence the ○^{s} whose centres are C and C′, and radii CA and C′A respectively, will satisfy the req^d conditions

Now $OA = \sqrt{2^2 + 2^2} = 2\sqrt{2}$, $OP = CP = r$, and $OC = 2\sqrt{2} - r$

But $OC^2 = OP^2 + CP^2$, $(2\sqrt{2} - r)^2 = r^2 + r^2$,

whence $r^2 + 4r\sqrt{2} - 8 = 0$

5 Let the pts in the given order be A, B, C and D Let AB and CD intersect in P Then it is easy to prove that OP bisects the ∠APD, and that the figure is symmetrical about OP From O draw ON perp to CD, and bisect the ∠PON by OQ meeting PD in Q From Q draw QK par^l to ON and meeting OP in K Then ∠KQO = ∠NOQ [*Th* 14] = ∠KOQ [*Const^n*] KQ = KO By symmetry, or by congruent $\triangle^s$ [*Th* 17], KQ is the perp from K on AB
the ○ whose centre is K, and radius = KO, is the req^d ○

6 Let ABC be the given equilateral $\triangle$; and AD, BE, CF the three medians, meeting at O. Then, by Theor. 7, AD bisects $\angle$BAC and is perp. to BC.

Bisect $\angle$OFB by FP meeting OB at P. Then since P lies on the bisectors of each of the $\angle$s OFB, FBD, FOD, a circle can be described with centre P to touch each of the sides of the quad[l] BFOD. [*Prob.* 26.]

Along OA, OC cut off OQ, OR each equal to OP.

Then from P, Q, and R as centres it follows by symmetry that equal circles can be drawn, each touching two sides of the $\triangle$ ABC and the other two circles.

Let M be the foot of the perp. from P on AB.

Then $FM=PM=r$; and $BM=PM \tan BPM=r \tan 60°$.

Now $FB=\frac{1}{2}AB=\frac{3}{2}$; and $FB=FM+BM$; $\therefore$ $r(1+\tan 60°)=\frac{3}{2}$.

Also $\tan 60°=\sqrt{3}$; $\therefore$ $r=\dfrac{3}{2(\sqrt{3}+1)}=\dfrac{3(\sqrt{3}-1)}{4}=0{\cdot}55''$.

7 Let OA, OB, OC be three radii each inclined at 120° to the others. Join AB.

Then it may be proved that AB is perp. to CO produced.

Bisect $\angle$OAB by AD meeting CO produced in D, and through D draw DP, par[l] to AB, to meet OA in P. Then $\angle$ODP is a rt. $\angle$.

And $\angle$PDA$=\angle$DAB [*Th.* 14]$=\angle$PAD; whence PA=PD.

Hence a $\odot$, centre P and radius PA, will touch the given $\odot$ and also CO produced at D.

Along OB, OC mark off OQ, OR each equal to OP.

It is now easy to shew that P, Q, R are the centres of the req[d] $\odot$s.

$OP=PD \operatorname{cosec} POD=r \operatorname{cosec} 60°$; and $OA=OP+PA$.

$$\therefore\ 2=r \operatorname{cosec} 60°+r,$$

$$\text{or}\ \ r(1+\operatorname{cosec} 60°)=2.$$

Now $\operatorname{cosec} 60°=\dfrac{2}{\sqrt{3}}$; $\therefore$ $r=\dfrac{2\sqrt{3}}{2+\sqrt{3}}=2\sqrt{3}(2-\sqrt{3})$

$$=4\sqrt{3}-6$$

$$=0{\cdot}93'', \text{ nearly.}$$

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1 Take one of the given sides as base, and from an extremity of this base, with the other side as radius, describe a ○

Then the vertex of the △ must lie on this ○ Now, the base being given, the △ is greatest when the altitude is greatest this may readily be shewn to be when the second side is drawn at rt angles to the first [*Th* 10]

2 If the base AB and area of a △ABC are given, the vertex C must move on a st line PQ par[1] to the base AB [*Th* 27]

And since AB is fixed, the perimeter is least when the sum of AC, BC is least This is the case when

the ∠PCA = the ∠QCB [*Ex* 3, p 316]

But ∠PCA = ∠CAB, and ∠QCB = ∠CBA [*Th* 14]

∠CAB = ∠CBA, AC = BC [*Th* 6]

If AB = 2 0″, and area = 3 12 sq in,

$$\text{the altitude} = \frac{\text{area of } \triangle}{\frac{1}{2}\text{ base}} = \frac{3\ 12}{1\ 0} = 3\ 12''$$

since the △ is isosceles,

$$AC^2 = 1^2 + (3\ 12)^2 = 10\ 7344 \text{ sq in}$$

whence $AC = \sqrt{10\ 7344} = 3\ 276''$

perimeter (when minimum) = 8 55″ approx

3 Since the base BC and the vertical angle are given, the vertex A must move on the segment of a ○ described on BC to contain an angle of 60° [*Th* 39 *Conv*]

Also since the base is given, the △ is greatest when the altitude is greatest, which is when A lies on AX the perpendicular bisector of BC

For take any other pt P on the arc and draw PM perp to BC meeting the diameter par[1] to BC at N The centre O lies on AX

Then OP > PN [*Th* 10], OA > PN

AX > PM, for OX = NM

Since AB = AC [*Th* 4], and ∠BAC = 60°, the △ must be equilateral

BA = BC = 10 cm Hence, by Theor 29, $AX = 5\sqrt{3}$ cm

area = $\frac{1}{2}$BC AX = $25\sqrt{3}$ sq cm = 43 30 sq cm

4 Bisect AB at C, and from C draw CM, CN perp to OX, OY respectively Then CM is par[1] to OB, and by Theor 22, $OM = \frac{1}{2}OA = 1\ 5''$ Thus M lies on the ○, and CM is a tangent Similarly CN is a tangent Then ∠MCN will be the maximum angle contained by tangents drawn from pts on AB

For take any other pt P on AB and draw the tangents PQ, PQ′ Then $OP > OC$ [*Th* 12]

∴ $OQ^2 + QP^2 > OM^2 + MC^2$ [*Th* 29], whence $QP > MC$

Now apply the △OQP to the △OMC so that OQ and OM coincide and QP lies along MC Then P lies in MC produced and ∠OCM > ∠OPQ

the ∠MCN (which is double ∠OCM) is greater than ∠QPQ′ (which is double the ∠OPQ)

By measurement ∠MCN is found to be a rt angle, which was to be expected since OMCN is a par[m], and ∠MON is a rt angle

5 Let AB be the straight rod, C its middle point, and O the intersection of the rulers Then OC = half of AB, and is constant for all positions of AB [*Th* 41]

And since the base AB is given in magnitude, the area of the △ is greatest when the perp from O on AB is the greatest

Now when AB makes equal angles with the two rulers, OC is perp to AB [*Th* 6 and 4] And in any other position of AB, OC is greater than the perp from O on AB [*Th* 12]

Hence the greatest △ is obtained when AB is equally inclined to the two rulers

6 Let AB be the given line, and K the side of the given square

(1) At B draw BC, making the ∠ABC half a rt angle

From centre A, with radius K, describe a ○ cutting BC at P (or P′) Draw PX perp to AB

Then, $AX^2 + XB^2 =$ sq on K

For ∠XBP = $\frac{1}{2}$ rt angle, and ∠PXB = one rt angle,

∠XPB = $\frac{1}{2}$ rt angle, PX = XB

Hence $AX^2 + XB^2 = AX^2 + XP^2$

$= AP^2 =$ sq on K [*Th* 29]

(ii) Thus AX^2+XB^2 is a minimum, when AP is a minimum, that is, when AP is the perp on BC

In this case the $\angle PAB=\frac{1}{2}$ rt angle $=\angle ABP$

AP=BP, and hence X is the middle point of AB

7 (i) Let C, D be the centres of the given ⊙s which intersect at A, and X the given line

On CD describe a semicircle, and from centre D with radius equal to half of X cut this semicircle at E Join ED

Through A draw PAQ par[l] to ED Then PQ is the line required. Join CE and produce it to meet PQ at G, and draw DH par[l] to CG, meeting PQ at H

Then since $\angle CED$ is a right $\angle$, [*Th* 41]

CG, DH are perp to PQ [*Th* 14], and $GH=ED=\frac{1}{2}X$

Also $PQ=2GH$, [*Th* 31]

$PQ=X$, and is drawn through A

(ii) We see that PAQ is a maximum, when ED is a maximum But ED has its greatest value when it coincides with CD Hence PQ is a maximum when it is par[l] to CD

8 (i) Let OA, OB be the tangents Take P the middle point of the major arc AB, and let PX, PY be the perps. on OA, OB

To prove that PX+PY is a maximum

Let Q be any other point on the arc AB (in the Fig chosen, Q is on PB), and QM, QN the perps on OA, OB

Let PY, QM intersect at R Join PQ, OP

Then since the tangent at P is perp to OP, and Q is on the ⊙ce,

the $\angle OPQ$ is less than a rt $\angle$

But $\angle RPO=\angle POA=45°$, $\angle RPQ$ is less than 45°,

, from the rt angled $\triangle QRP$, the $\angle RQP$ is greater than 45°,

$\angle RPQ$ is less than $\angle RQP$

RQ is less than RP [*Th* 10]

Also $RM=PX$, and $QN=RY$

$RQ+RM+QN$ is less than $RP+PX+RY$,

or $QM+QN$ is less than $PX+PY$

(ii) Similarly if P′ is the middle point of the minor arc AB, it can be proved that P′X+P′Y is a minimum

Let C be the centre of the ○, then OACB is a square and CP=CB=2 0″

Let PX meet the diam par¹ to OX in H

Then it is easily shewn that OX=PX and CH=PH

But $2CH^2 = CP^2$ [*Th* 29] = 4 sq in   CH = $\sqrt{2}$ in

$$OX = OA + AX = (2+\sqrt{2}) \text{ in} = 3\ 414''$$

$$OX + PX = 6\ 83''$$

Similarly $OX' + PX' = (4 - 2\sqrt{2}) \text{ in} = 1\ 17'$

9. Let A and B be the fixed points, PQ the tangent at T, and let ∠PTA=∠QTB

To prove that AT+BT is a minimum

This problem supposes that AB does not meet the ○, and that AT, BT are on the side of PQ remote from the ○.

Take X any other point on the Oce then AX must cut PQ [*Hyp*] at some point K Join KB, XB

Then AK+KB is greater than AT+TB, [*Ex* 3, p 316]

and AX+XB is greater than AK+KB [*Ex* 3, p 34]

Hence AX+XB is greater than AT+TB

10 Let AP, AQ be st lines of indefinite length including the fixed vertical angle

Let ABC be the isosceles △, having the given altitude AD

And let AB′C′ be any other △ having an equal altitude AD′

Then by Theor 17, D is the middle point of BC

Through D draw XDY par¹ to B′C′ meeting AP, AQ at X, Y

Now △ABC is less than △AXY [*Ex* 4, p 316]

And △AXY is less than △AB′C′ by the strip B′XYC′, since it may be shewn that B′C′ must lie on the side of XY remote from A For let XY meet AD′, or AD′ produced, at K then the ∠AKD is a rt angle, ∴ AD is greater than AK, that is, AD′ is greater than AK

When the given vertical angle is 60°, for minimum area each of the base angles must be also 60° [*Th* 16], and the triangle is equilateral Let $2a$ be the length of each side

Then by Th 29, $4a^2 = a^2 + 36$, whence $a = 2\sqrt{3}$ cm

minimum area $= \frac{1}{2}(6 \times 4\sqrt{3})$ sq cm $= 20\ 78$ sq cm (to nearest sq mm),

and perimeter of minimum $\triangle = 12\sqrt{3}$ cm $= 20\ 8$ cm (to nearest mm)

11 Let AP and AQ be the two fixed tangents, and BC any other tangent to the convex arc Let O be the centre of the ○, D the pt of contact of BC Join OB, OC, OD, OP, OQ.

Then as in Th 47, Cor, the △ˢ BOP, BOD are congruent, and the △ˢ COD, COQ are also congruent

Hence the fig BPOQC $= 2\triangle$ BOC, and $\angle$POQ $= 2\angle$BOC

Now the quadˡ APOQ is of constant area, and the △BAC is a maximum when the fig BPOQC is a minimum, that is, when △BOC is a minimum

But $\angle$BOC $= \frac{1}{2}\angle$POQ $=$ a constant, and the altitude OD of the △BOC is constant

△BOC is a minimum when it is isosceles [*Ex* 10], that is, when BC touches the arc PQ at its middle point

12 Take AB $= 1\ 6''$ as base then the altitude $= \frac{1\ 2}{0\ 8} = 1\ 5''$

C the vertex lies on a line XY parˡ to AB and distant $1\ 5''$ from AB draw this line and describe a circle to pass through A, B, and touch XY at C [*Ex* 2, p 311] Then the $\angle$ACB will be the greatest vertical angle, and on measurement it is found to be 56° (to the nearest degree) Proof by Ex 2, p 315

13 Let A, B be the given points (both without the given ○) Through A and B describe a ○ to touch the given ○ externally at C [*Ex* 3, p 312]

To prove that ACB is the maximum angle Let P be any other point on the ○ᶜᵉ of the given ○ Join AP, BP, and let AP meet the ○ of construction at Q Join QB

Then $\angle$AQB is greater than $\angle$APB, [*Th* 8]

and $\angle$AQB $= \angle$ACB [*Th* 39]

$\therefore$ $\angle$ACB is greater than $\angle$APB

If A, B are both within the given ○, the ○ of construction must be drawn so as to have internal contact Here there are always two solutions, and the one which gives the greater angle must be chosen

14 Draw a ○ to pass through A, B, and touch OY [*Ex* 2, p 311]
Let P be the pt of contact Then ∠APB will be the required maximum angle [*Ex* 2, p 315]

Now $OP^2 = OA \cdot OB = 0.8 \times 1.8 = 1.44$ sq in [*Th* 58]

$OP = 1.2''$

15 Let ABCD represent the bridge, where AB=49 ft, BC=32 ft, and CD=49 ft The st line AP represents the bank

Through B, C describe a ○ to touch AP at T [*Ex* 2, p 311]

Then the arch BC subtends the greatest angle at T [*Ex* 2, p 315]

Also $AT^2 = AB \cdot AC$ [*Th* 58]

$= (49 \times 81)$ sq ft

$AT = 7 \times 9 = 63$ ft

16 Since the sides AC, BC are constant, the area of the △ACB is a maximum when they are at rt angles [*Ex* 1, p 317]

Draw any two radii CA′, CB′ at right angles, and join A′B′

With centre C and radius equal to the perp from C on A′B′ describe a ○, and let the tangent from P to this ○ cut the given ○ in A and B Then AB=A′B′ [*Th* 34], and hence by Theor 7 the ∠ACB is a rt angle

If the radius=6 cm,

the area of $\triangle ACB = \frac{1}{2} AC \cdot CB$ (since $\angle ACB = 90°$)

$= \frac{1}{2}(6 \times 6)$ sq cm

$= 18$ sq cm

17. Let ABCD be a rectangle inscribed in the given ○

Join AC Then AC is a diameter [*Th* 41]

Now the rectangle is double of the △ABC

And since the base AC is constant, the △ABC is greatest when the altitude BX, namely the perp from B on AC, is greatest

And BX may be shewn to be greatest when B is the middle point of the arc AC The rectangle then becomes a square

And if the radius of the circle be 5.5 cm,

area of the rect $= 2\triangle ABC = AC \cdot BX = 11 \times 5.5 = 60.5$ sq cm

18 Let O be the centre of the given ○ Bisect AB at X, and join XO cutting the ○^{ce} at P Join AP, PB

To prove that AP^2+PB^2 is a minimum

Now $AP^2+PB^2=2AX^2+2XP^2$ [*Th* 56]

Hence, since AX is constant, AP^2+PB^2 is a minimum when XP^2 is a minimum

But XP is the least of all st lines drawn from X to the ○^{ce} [*Th* 37]

19 It is shewn in Ex 4, p 191, how to find a point C such that AC+BC may be equal to a given line H Now the greatest value H can have, in order that this construction should be possible, is the diameter of the second segment there employed This determines the point X, and therefore the point C and it may easily be shewn by Theor 41 that CX=CB=CA, that is, that C is the middle point of the arc AB

20 No inscribed triangle that is not equilateral can have the maximum perimeter

For let PQR be an inscribed triangle not equilateral, then it must have one pair of sides unequal, say PQ, QR Hence there is an inscribed △ on the base PR, which has a greater perimeter [*Ex* 19], the △PQR is not the inscribed △ of greatest perimeter And this argument may be applied to *all* inscribed triangles not equilateral

21 No inscribed triangle that is not equilateral can have the greatest area

For let PQR be an inscribed △ not equilateral, then it must have one pair of sides unequal, say PQ, QR Hence [as in Ex 3, p 317] there is an inscribed △ on the base PR, which has a greater area

the △PQR is not the inscribed △ of greatest area

And this argument may be applied to *all* inscribed △^{s} not equilateral

22 It has been proved [II p 208] that every two sides of the pedal triangle are equally inclined to that side of the original triangle in which they meet

Also [*Ex* 3, p 316] if A and B are fixed points and P a point

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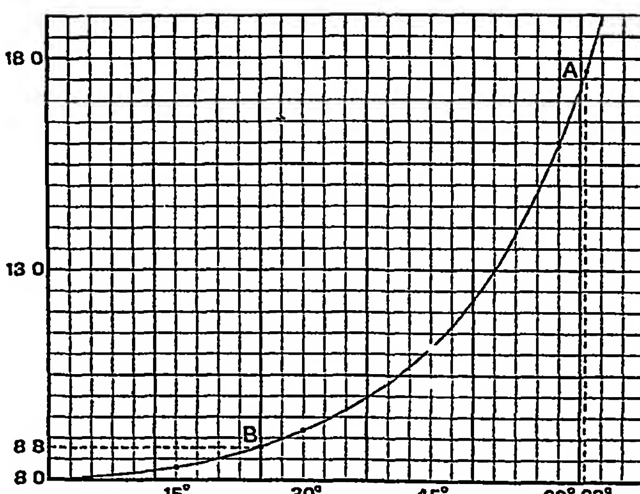
The scales used in the following solutions are chosen for convenience of representation on the printed page The student should in most cases choose his own so as to produce diagrams of about twice the size shewn

1 The tabulated results are

a	15°	30°	45°	60°	75°
PR	8 3	9 2	11 3	16 0	30 9

The scale adopted is that in which 2 horizontal divisions represent 5° and 2 vertical divisions represent 1 cm We then obtain the adjoining graph [The portion beyond 65° has been cut off from considerations of space]



The ordinate of A, viz 17 6 cm, and the abscissa of B, viz. 25°, give the values required

2 By actual measurement of the perpendiculars from B on AC corresponding to different values of C we obtain the following tabulated results.

Angle C	0°	30°	45°	60°	90°	120°	135°	150°	180°
Altitude in cm	0	2	2·83	3·46	4	3·46	2·83	2	0
Area in sq cm	0	5	7·07	8·66	10	8·66	7·07	5	0

Now proceed as in Ex. 2. p 321, taking one horizontal division to every 6°, and 2 vertical divisions to every sq cm

The graph presents no difficulty, and it will be found that when C is 63° the area is 8·9 sq. cm and that when the area is 9·5 sq cm the angle is either 72° or 108°. Also the maximum area is 10 sq cm., when $C=90°$.

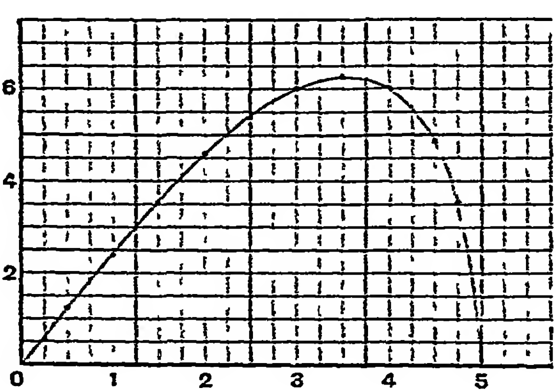
3. Let $CA=x$ cm., then, by Theor 29, $CB=\sqrt{25-x^2}$

area of $\triangle ABC=\frac{1}{2}$ AC CB $=\frac{1}{2}x.\sqrt{25-x^2}$.

Hence by substituting different values for x we obtain

x	0	0·5	1·0	1·5	2·0	2·5	3·0	3·5	4·0	4·5	5·0
Δ	0	1·24	2·45	3·58	4·58	5·41	6·0	6·25	6·0	4·90	0

Taking 4 horizontal divisions for the x unit and 2 vertical divisions for the Δ unit, we obtain the following graph



It is shewn in Ex 5, p 317 that Δ is a maximum when the rod is equally inclined to the two rulers; and, by actual

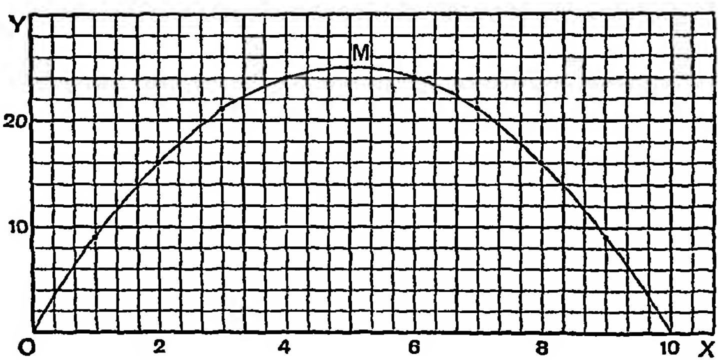
calculation from Theor 29, or by use of a larger graph, it will be found that this occurs when CA=3 53 cm

4 (i) Let AP=x cm, PB=10−x Hence denoting AP PB by y we must draw the graph of $y=x(10-x)$

The following values may be used

x	0	1	2	3	4	5	6	7	8	9	10
y	0	9	16	21	24	25	24	21	16	9	0

Hence taking 3 horizontal divisions as the unit of x and 1 vertical division to represent *two* units of y we obtain the following graph



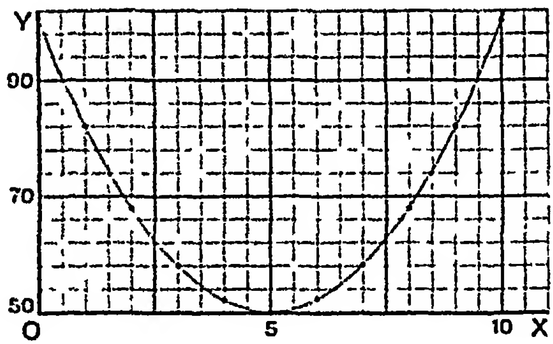
It is seen that y has its maximum value at the point M whose abscissa is 5 Hence AP PB is a maximum when AP is 5 cm, that is, when P is at the mid-pt of AB

(ii) Here $y=AP^2+PB^2=x^2+(10-x)^2=100-2x(10-x)$

Use the following values

x	0	1	2	3	4	5	6	7	8	9	10
y	100	82	68	58	52	50	52	58	68	82	100

Taking 2 horizontal divisions as 1 unit of x, and 1 vertical division as 2 units of y we have the adjoining graph

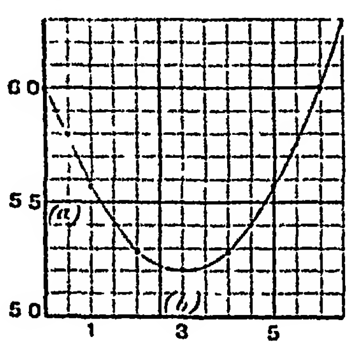


It is seen that y has its minimum value at the point whose abscissa is 5 Thus AP^2+PB^2 is a minimum when AP=5 cm, that is, when P is at the mid pt of AB

5 Draw an angle BAC=60° with AB=6 cm By taking different values of b along AC and measuring the corresponding lengths of BC (=a) we obtain the following table

b	0	1	2	3	4	5	6	7	
a	6	5 57	5·29	5 20	5 29	5 57	6 0	6 56	

Taking 2 horizontal divisions to represent the unit of b, and 10 vertical divisions to represent the unit of a we obtain the adjoining graph

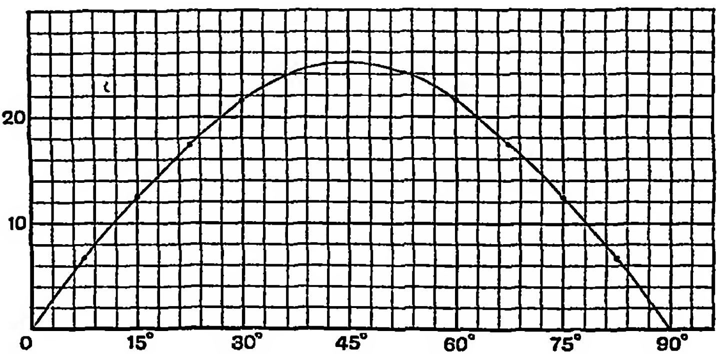


The minimum value of a is seen to be 5 2, and on drawing the triangle for this value we find that BC is perpendicular to AC as we should expect from Theor 12

6 Take AB = 10 cm and measure the altitude (p) of the Δ PAB for different values of the $\angle$PAB Then since the area $(\Delta)=\frac{1}{2}\ p$ AB $=5p$, we have the following results for p and Δ, in cm and sq cm respectively

$\angle$PAB	0	$7\frac{1}{2}°$	15°	$22\frac{1}{2}°$	30°	$37\frac{1}{2}°$	45°	$52\frac{1}{2}°$	
p	0	1 29	2 5	3 53	4 33	4 83	5 0	4 83	
Δ	0	6 45	12 5	17 65	21 65	24 15	25 0	24 15	

the values, after those corresponding to 45°, recurring in reverse order



The graph is then as shewn, the scales being 1 horizontal division = 3°, and 1 vertical division = 2 sq cm, and it is seen that Δ is maximum when the angle is 45°

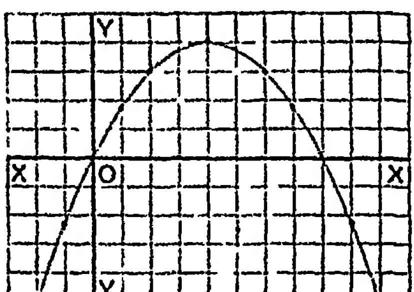
7 If we take y to represent the area of any one of the figures and x to represent the length of the side on which it is placed we know from Theor 73 that the ratio y x^2 is the same for all the figures of the series Thus the fraction $\frac{y}{x^2}$ is constant and its value (which may be found if required by calculation from any one of the figures) can be denoted by m Hence if we draw the graph of $\frac{y}{x^2}=m$, or $y=mx^2$, the ordinate corresponding to any abscissa will give the area of the figure whose side is equal to that abscissa,

If the given figures are squares, $m=1$ for when x is 1 inch, y is 1 sq. in. Hence the graph to be drawn is that of $y=x^2$ which presents no difficulty. The abscissa corresponding to the ordinate 11.8 will be found to be 3.43.

[One inch on each axis will be found a convenient unit.]

8. (i) The graph is shewn in the diagram, the values being as tabulated

x	2	0	1	2	4	6	8	10
$2x$	−4	0	2	4	8	12	16	20
$\frac{x^2}{4}$	1	0	0.25	1	4	9	16	25
y	−5	0	1.75	3	4	3	0	−5

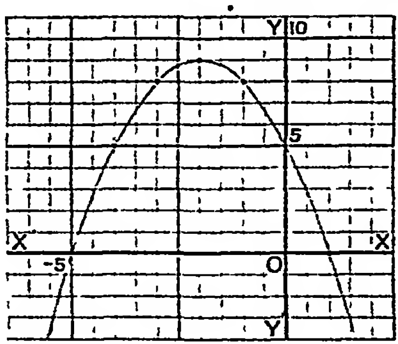


One division on each axis has been taken as unit.

(ii) Use the following values

x	−6	−5	−4	−3	−2	−1	0	1	2
$4x$	−24	−20	−16	−12	−8	−4	0	4	8
x^2	36	25	16	9	4	1	0	1	4
y	−7	0	5	8	9	8	5	0	−7

Taking 2 horizontal divisions as the unit of x and 1 vertical division for the unit of y we obtain the adjoining graph



The maximum value of y is given by the maximum ordinate which is seen to be 9

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1 (i) By Ex. 1, p 324, AX, AB, AY are in H P

then reciprocals $\frac{1}{AX}$, $\frac{1}{AB}$, $\frac{1}{AY}$ are in A P

$$\therefore \frac{2}{AB}=\frac{1}{AX}+\frac{1}{AY}$$

(ii) By the note to the Def p 324, YX is divided harmonically at B A

$\therefore$ as in (i), $$\frac{2}{YX}=\frac{1}{YA}+\frac{1}{YB}$$

2 (i) Here XB=AB−AX=2·4″−1·5″=0·9″; and Y divides AB externally in the ratio AX : XB, viz. 1·5 : 0·9, or 5 : 3

$$\therefore AY=\frac{5}{5-3}\cdot AB=6·0''$$ [*Ex* 7 p 253]

(ii) Here AX=AY−XY=0·5 cm, and since B, A divide YX harmonically,

we have YB : BX=YA : AX=2·0 : 0·5=4 : 1

$$\therefore YB=\frac{4}{4+1}\cdot YX=1·2 \text{ cm}$$

3 See Theor 61

4 Let A be any pt at which BP, PC subtend equal angles. Divide BC externally at Q in the ratio BP : PC, and join AB, AC, AP, AQ.

Because ∠BAP = ∠CAP [*Hyp.*], BA : AC = BP : PC [*Th.* 61.]
= BQ : QC [*Constⁿ.*]

∴ AQ bisects the ∠BAC externally [*Th.* 61.]

∠PAQ is a rt. ∠.

But P, Q are fixed pts. ∴ locus of A is the circle on PQ as diameter.

5. Let the line bisect the base BC at K, cut AB produced in P, and AC in Q, and the par.¹ to BC through A in M.

The △ˢ KQC, MQA are similar by Theors. 14, 62,

QK : QM = CK : AM
= BK : AM [*Hyp.*]
= PK : PM (from similar △ˢ PBK, PAM)

P and Q divide KM harmonically.

6. Let a line through B meet the median AD in K, and cut AC in Q, and the par.¹ to BC through A in M.

Through K draw XKY par.¹ to BC and meeting AB, AC in X and Y.

Then by similar △ˢ, XK : BD = AK : AD = KY : DC
[*Th.* 14 and 62.]

Hence XY is bisected at K, and the proof proceeds as in Ex. 5.

7 Through P draw XPY par.¹ to CQ, meeting AC in X and CB produced in Y.

From the similar △ˢ APX, AQC, we have AP : AQ = PX : QC, and from the similar △ˢ PBY, QBC we have

BP : BQ = PY : QC

But AP : AQ = BP : BQ [*Hyp.*]

PX : QC = PY : QC, whence PX = PY.

Now ∠CPX = ∠PCQ = rt. ∠ [*Th.* 14.]

the △ˢ CPX, CPY are congruent [*Th.* 4.]

∠PCX = ∠PCY.

Hence CP bisects ∠ACB internally, and CQ, being perp. to CP, bisects the ∠ACB externally.

8 $OX \; OY = OB^2$ [*Ex* 2, p 324]

Hence the product OX OY is always constant

When X is very near O, OX is very small, and OY is very large, Y being infinitely remote when X coincides with O

As OX increases, OY diminishes and when OX = OB, OY = OB

Thus as X moves from O to B, Y moves from an infinite distance towards B, ultimately coinciding with X at the pt B

Denoting the lengths of OX, OY by x and y, we have $xy = 100$, and the resulting graph is a *rectangular hyperbola* whose asymptotes are the axes of coordinates

9. Through P and C draw PN, CM each par¹ to BD to meet AB in N and M

Then AB − PQ PQ − CD = AN NM [*Th* 21]

= AP PC [*Th* 60]

= AO OD [*Th* 60]

= AB CD (from the similar △ˢ AOB, DOC)

PQ is the harmonic mean between AB and CD [*Def* 3, p 323]

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1 (i) Let A, P, B, Q be a harmonic range, and S the vertex of the pencil Through P draw aPb par¹ to SQ meeting SA, SB at a and b

Now AP PB = AQ QB [*Hyp*]

Alternately AP AQ = PB QB

But from the similar △ˢ APa; AQS

AP AQ = aP SQ,

and from the similar △ˢ BPb, BQS

PB QB = bP SQ,

aP SQ = bP SQ,

aP = bP

Hence, as in Ex 3, p 263, it may be shewn that *any* transversal $a'p'b'$ par¹ to aPb (that is, par¹ to SQ) has equal parts intercepted by the rays SA, SP, SB

(ii) Conversely, let the pencil be cut by a transversal $a'p'b'$ par[1] to SQ, so that $a'p'=b'p'$ it will be proved that the pencil is harmonic

As before, through P draw aPb par[1] to $a'p'b'$ (or SQ) Then from the similar $\triangle^s$ APa, AQS ,

$$\text{AP}\quad \text{AQ}=a\text{P}\quad \text{SQ}.$$

And from the similar $\triangle^s$ BPb , BQS,

$$\text{PB}\quad \text{QB}=b\text{P}\quad \text{SQ}.$$

But since $\quad a'p'=b'p'$ [*Hyp*] $\qquad a\text{P}=b$

$$a\text{P}\quad \text{SQ}=b\text{P}\quad \text{SQ}.$$

Hence $\qquad \text{AP}\quad \text{AQ}=\text{PB}\quad \text{QB}$

Alternately $\qquad \text{AP}\quad \text{PB}=\text{AQ}\quad \text{QB},$

or, A, P, B, Q is a harmonic range

NOTE As a second converse it may be shewn indirectly that if the range is harmonic, and if in any transveral $a'p'=b'p'$, then $a'p'b'$ is par[1] to SQ.

2 Let a harmonic pencil be formed by joining a point S to the harmonic range A, P, B, Q , then it will be proved that any transversal is cut harmonically by this pencil

Through P draw *any* transversal aPbq, and also the transversal hPk par[1] to SQ.

Then by Ex 1 (i), $\qquad h\text{P}=k\text{P}$

Hence by Ex 1 (ii) the range a, P, b, q is harmonic , any transversal $a'p'b'q'$ par[1] to aPbq is also cut harmonically [See *Ex* 3, p 263]

3 (i) In the harmonic pencil {S, APBQ} let one ray SP bisect the angle between the rays SA, SB , it will be proved that SP is perp to SQ.

Through P draw aPb par[d] to SQ , then since the pencil is harmonic, $a\text{P}=b\text{P}$ [*Ex* 1]

Also in the $\triangle a\text{S}b$, since SP bisects the vert $\angle$, and $a\text{P}=b\text{P}$,

$$a\text{S}=b\text{S} \qquad [\textit{Th } 61]$$

Hence the $\triangle^s$SPa, SPb are identically equal, so that ab is perp to SP

also SQ is perp to SP $\qquad$ [*Th* 14]

(ii) *Conversely*, in the harmonic pencil {S, APBQ} let PSQ be a rt angle, it will be proved that SP, SQ are the internal and external bisectors of the angle ASB

As before, draw aPb par[1] to SQ, then $aP=Pb$ [*Ex* 1], and the $\angle^s$ SPa, SPb are rt angles, [*Th* 14]

hence the Δ^sSPa, SPb are identically equal, [*Th* 4]

the $\angle a$SP$=$the $\angle b$SP

That is, SP is the internal bisector of the $\angle$ASB, and since SQ is perp to SP, SQ is the external bisector

4 Join SQ cutting the transversal $apbq$ in q'

Then {S, APBQ} is a harmonic pencil by definition, hence, a, p, b, q' is a harmonic range [*Ex* 2],

but by hypothesis a, p, b, q is a harmonic range,

the points q, q' coincide, since they divide ab externally in the fixed ratio ap pb [*Th* vi p 252]

Hence SQ passes through q, or Qq passes through S

5 Let Pp, Bb, produced if necessary, meet at S Join SA, SQ, and let SQ meet the transversal Apb at q'

Then, as in the last example, {S, APBQ} is a harmonic pencil,

A, p, b, q' is a harmonic range [*Ex* 2]

But A, p, b, q is a harmonic range, [*Hyp*]

q and q' are coincident, or, Qq passes through S,

that is, Pp, Bp, Qq are concurrent

Similarly it may be shewn that Pq, Bb, Qp are concurrent

6 *Lemma* Take two straight lines intersecting at A, and in one of them take *any* two points P, Q, and in the other any two points p, q Let Pp and Qq intersect at S, and Pq, Qp at S', now it is proved in Ex 5, that if B and b are the harmonic conjugates of A with respect to P, Q and p, q respectively, then B, b will lie on the fixed line SS' Hence it follows, if SS' intersects the given lines at B, b, that A, P, B, Q and A, p, b, q are harmonic ranges

Now let PQqp be a quadrilateral, and let the sides QP, qp meet at A, and the sides Pp, Qp at S A *complete quadrilateral* will then be formed

Let the diagonals Pq and Qp intersect at S' then if SS' meets PQ at B, the range A, P, B, Q is harmonic

Let the diagonals Pq, Qp meet the third diagonal SA at X and Y it is required to shew that SA is cut harmonically at X and Y Join $S'A$

Then $\{S', APBQ\}$ is a harmonic pencil, therefore it cuts any transversal (such as the third diagonal AS) harmonically That is, the range A, X, S, Y is harmonic

NOTE The Lemma attached to this proposition furnishes a simple *linear* construction for finding a fourth harmonic to three points

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1. Draw any radius CP in the first O, and a par[l] diam[r] $P'C'P''$ in the second O, and measure the distance between the points where PP', PP'' cut the line of centres

Suppose S and S' to be these pts, S being the external pt

Then $\quad CS \; SC' = r \; r' = 3 2 \; 1 2 = 8 \; 3,$

and $\quad CS' \; S'C' = r \; r' = 8 \; 3$

Hence S, S' divide CC' externally and internally in the ratio 8 3

$$CS = \frac{8}{8-3} \text{ of } 5\ 5 \text{ cm} = 8\ 8 \text{ cm}$$

and
$$CS' = \frac{8}{8+3} \text{ of } 5\ 5 \text{ cm} = 4\ 0 \text{ cm}$$

2 With the fig of p 329, taking $CS = 2\ 7''$, we have

$$C'S \; SC = r' \; r = 1\ 0 \; 1\ 8 = 5 \; 9$$

$$C'S = \tfrac{5}{9} \text{ of } 2\ 7'' = 1\ 5''$$

$$CC' = CS - C'S = 1\ 2''$$

Also
$$CS' \; S'C' = r \; r' = 9 \; 5$$

$$CS' = \tfrac{9}{14} \text{ of } 1\ 2'' = 0\ 77''$$

$$SS' = CS - CS' = 1\ 93''$$

3 From the similar Δ^s SCP, SC′P′, we have

$$SP \;\; SP' = CP \;\; C'P = r \;\; r'$$

Also since CT, C′T′ are par[1], the Δ^s SCT, SC′T′ are similar,

$$ST \;\; ST' = CT \;\; C'T' = r \;\; r'$$

$$SP \;\; SP' = ST \;\; ST'$$

Hence PT is par[1] to P′T′ [*Th* 60]

$\angle SPT' = \angle SPT = \angle STQ$ [*Th* 49]

T, Q, P, T′ are concyclic [*Ex* 5, p 163]

$SP' \;\; SQ = ST \;\; ST'$ [*Th* 58]

Similarly $SP \;\; SQ' = ST \;\; ST'$

$SP \;\; SQ' = SP' \;\; SQ = ST \;\; ST'$

4 See Fig, p 213 Here AB, AC produced are common tangents, A is one centre of similitude, and since BI, BI_1 are the internal and external bisectors of the $\angle$ ABC, the pencil $\{B, AIYI_1\}$ is harmonic. [*Th* 61 *Cor*]

$r \;\; r_1 = IA \;\; I_1A = IY \;\; I_1Y$

Y is the other centre of similitude

5 Taking the Fig and the results of p 217, we have from the similar Δ^s ASO, aNO,

$SO \;\; NO = SA \;\; Na$

$=$ circum-radius nine-points-radius

O is the external centre of similitude of the two circles [*Ex* 1, p 328]

Again from the similar Δ^s ASG, XNG,

$SG \;\; GN = SA \;\; NX$

$=$ circum-radius nine-points-radius

G is the internal centre of similitude of the two circles

6 Let C, C′ be the centres of the two fixed $\odot^s$ external to one another, and O the centre of a variable $\odot$ touching the others at P, Q respectively In the Fig taken, the given $\odot^s$ are both external to the $\odot$ (O) Then OC, OC′ pass respectively through P and Q [*Th* 48]

Produce PQ to cut the $\odot$ (C′) at P′, and join C′P′

Then, since OP = OQ, and C′P′ = C′Q,

$\angle OPQ = \angle OQP =$ vert opp $\angle C'QP' = \angle C'P'Q$

CP and C′P′ are par[1]

PP passes through the external centre of similitude S

It will be found that if the given ⊙s are both external, or both internal, to the variable ⊙, then PQ passes through the *external* centre of similitude

If one of the given ⊙s is within, and the other without the variable ⊙, it will be found that PQ passes through the *internal* centre of similitude

7. Let C, C′ be the centres of the given circles, and X the given point

Take S the external centre of similitude, and let CC′S cut the given ⊙s between C, C′ at M and N

Join SX, and in SX (by describing a ⊙ through MNX) take a point Y, such that

$$SX \cdot SY = SM \cdot SN$$

By Ex 3, p 312, describe a ⊙ to pass through X, Y and to touch the ⊙ (C) at P This ⊙ will be proved also to touch the ⊙ (C′) Let O be its centre

Let SP, produced if necessary, meet the ⊙ (C′) at Q

then $SX \cdot SY = SM \cdot SN$ [*Constr*] $= SP \cdot SQ$ [*Ex* 3, p 329]

∴ the ⊙ (O) passes through Q [*Ex* 6, p 236]

It remains to prove that (O) *touches* (C′) at Q, that is, that OQ, C′Q are in one line Let SPQ meet the ⊙ (C′) again at P′, then since P, P′ are corresponding points, CP is par^l to C′P′ hence

$$\angle OQP = \angle OPQ = \text{alt } \angle C'P'Q = \angle C'QP',$$

but PQP′ is one st line, ∴ OQC′ is one st line

Since two ⊙s can be dr[illegible] through X, Y to touch the ⊙ (C) [*Ex* 3, p 312], it [illegible] that there are two solutions of the problem corr[illegible]ing to the external centre of similitude Similar[illegible] there will be two more solutions corresponding to the internal centre of similitude

8 Let A, B, C be the centres of the given ⊙s

Take the general case when the ⊙s are unequal and external to one another Let (A) be the least of the given ⊙s From centre B, with radius equal to the difference of the radii of (B) and (A) describe a ⊙, and from centre C with radius equal to the difference between the radii

of (C) and (A), describe a ○ Then by the last exercise describe a ○ to pass through A and to touch the two ○s of construction Take O the centre of the last drawn ○, and join OA, cutting the ○ (A) at P Then a ○ described from centre O with radius OP will touch the three given ○s The validity of this construction is apparent at once on drawing the figure

As each of the given ○s may be touched by the required ○ either internally or externally, the required ○ may in general be drawn in $2 \times 2 \times 2$, or 8, ways

The student will have no difficulty in investigating special cases for himself

9 Let r_1, r_2, r_3 be the radii of the three ○s

(i) Let $S_2'C_2$, $S_3'C_3$ intersect in O, join C_1O and produce it to meet C_2C_3 at X

The $\triangle$s C_2OC_3, C_3OC_1 are on the common base OC_3, and it may be proved by similar triangles that

the alt of $\triangle C_2OC_3$ the alt of $\triangle C_3OC_1 = C_2S_3'$ C_1S_3'

$$\frac{\triangle C_2OC_3}{\triangle C_3OC_1}=\frac{C_2S_3'}{C_1S_3'}=\frac{r_2}{r_1}$$

Similarly
$$\frac{\triangle C_1OC_2}{\triangle C_2OC_3}=\frac{C_1S_2'}{C_3S_2'}=\frac{r_1}{r_3}$$

, by multiplication, $\triangle C_1OC_2 \quad \triangle C_3OC_1 = r_2 \quad r_3$

But $\triangle C_1OC_2 \quad \triangle C_3OC_1 = C_2X \quad C_3X$,

$C_2X \quad C_3X \quad = r_2 \quad r_3$

X coincides with S_1', hence $S_1'C_1$, $S_2'C_2$ and $S_3'C_3$ are concurrent

(ii) To prove S_1, S_2', S_3' collinear

Join $S_2'S_3'$, and produce it to meet C_2C_3 at Y

Then C_2C_3, the external diagonal of the quad' $C_1S_2'OS_3'$ is divided harmonically at S_1' and Y [*Ex* 6, p 327]

hence Y, the harmonic conjugate of S_1' with respect to C_2C_3 is coincident with S_1, or S_1, S_2', S_3' are collinear

In the same way it may be shewn that each of the ranges of points consisting of one external and two internal centres of similitude are collinear, and also that the three external centres are collinear

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1 Let O and C be the centres, and P the pt of intersection of the two circles

Let PK be the tangent at P to the ○ (O) Then OP is perp to PK [*Th* 46], and is the tangent at P to the ○ (C) [Def of Orthogonal ○s]

Hence the tangent at P to the ○ (C) passes through O

Similarly the tangent at P to the ○ (O) passes through C

2. As in Ex 1, ∠OPC is a rt ∠ $OC^2 = OP^2 + PC^2$ [*Th* 29]

3 By Ex 1, the reqd locus is the tangent to the given ○ at the given point

4. Let Q be the pt through which the reqd ○ has to pass, P the pt at which it has to cut the given ○ orthogonally, and O the centre of the given ○

By the last Example the centre of the reqd ○ lies on the tangent at P to the given ○

By Prob 14, it must also lie on the perpendicular bisector of PQ

The centre of the reqd ○ is therefore determined

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1 Let A and B be the two given points, and let P be the intersection of their polars Then by the Reciprocal Property of Pole and Polar, since the polar of A passes through P,

the polar of P passes through A

Similarly, since the polar of B passes through P,

the polar of P passes through B

Hence the polar of P passes through both A and B, that is, AB is the polar of P

2 Let P be the intersection of the given st lines PQ, PR, and let A and B be their poles

Then since AB passes through A, ∴ its pole lies on PQ the polar of A

Similarly since AB passes through B, ∴ its pole lies on PR the polar of B

Hence the pole of AB is at P, the only point common to PQ and PR

3 The locus must be the polar of the given point A, for by the Reciprocal Property of Pole and Polar, (i) the pole of any st line through A must lie on the polar of A, and (ii) any point on the polar of A must be the pole of some st line through A

4 Let O be the common centre, P the point of contact of any one of the tangents, and Q its pole then since the tangent is perp to OP [*Th* 46], Q must lie on OP (or OP produced), and OP OQ = the sq on the radius of the given circle But this radius is constant, and OP is constant, ∴ OQ is constant Hence the locus of Q is a concentric circle

5 Let PQ be a diameter of one of the ⊙s, and let O be the centre, and r the radius of the other From O draw OT touching the first ⊙, and join OP cutting the first ⊙ at R Join QR

Now OR OP = OT² [*Th* 58]

= r^2, since the circles are orthogonal

and QRP is a rt ∠, being in a semicircle

Hence QR is the polar of P that is, the polar of P passes through Q

6 Let P and O be the centres of the two ⊙s which intersect at A, B and let OP cut AB at Q Join PA, PB

Then since the ⊙s are orthogonal, PA and PB touch the ⊙ (O) at A and B hence OP OQ = (radius)² [*Th* 66, *Cor*]

And OP meets the chord of contact at rt angles [*Ex* 7, p 177] ∴ AB is the polar of P with regard to the ⊙ (O)

7 Let A and B be the given points, and O the centre of the given ⊙ Then since the polars of A and B are respectively perp to OA, OB, ∴ one of the ∠s between the polars = the ∠AOB [*Ex* 8, p 43]

8 Let Q be the point inverse to P with respect to the given ⊙ Draw OY perp to AB, and through Q draw QX perp to OP meeting OY at X

Then since the ∠s at Q and Y are rt angles,

the points Q, X, Y, P are concyclic [*Th* 41, *Conv*]

$$OX \cdot OY = OP \cdot OQ \qquad [Th\ 58]$$

$$= r^2 \qquad [Hyp]$$

But OY is constant, ∴ OX is constant, that is, X is a fixed point

And since the ∠OQX is a rt ∠ [*Const*], ∴ the locus of Q is a circle on OX as diam [*Th* 41]

9 Let Q be the point on OP inverse to P, and r the radius of the ⊙ whose centre is O Draw OX a diam of the first ⊙ Join PX, and draw QY perp to OX

Then OPX is a rt ∠, being in a semicircle,

and QYX is a rt ∠ by construction,

∴ the points Q, Y, X, P are concyclic, [*Th* 41, *Conv*]

$$OX \cdot OY = OP \cdot OQ = r^2$$

But since OX is constant, ∴ OY is constant,

hence Y is a fixed point

Therefore the locus of Q is the st line perp to OX through the point inverse to X, that is, the polar of X

10 Let C and D be the points inverse to A and B respectively, and let AX, BY be the perps from A and B on the polars of B and A From A and B draw AM, BN perp respectively to OB and OA (produced if necessary)

Then $$OA \cdot OC = OB \cdot OD = r^2 \qquad [Definition]$$

And since the ∠s at M and N are rt ∠s, the points M, B, N, A are concyclic,

$$OA \cdot ON = OB \cdot OM \qquad [Th\ 58]$$

By subtraction

$$OA \cdot NC = OB \cdot DM$$

But NC = BY, and DM = AX [*Th* 21]

$$OA \cdot BY = OB \cdot AX$$

11 Let RQ cut AD and BC at p and p' Then it was proved in the solution of Ex 6, p 327, that the ranges P, A, p, D and P, B, p', C are harmonic

Hence by the *harmonic property* of Pole and Polar, the polar of P passes through both p and p' that is, RQ is the polar of P Similarly it may be shewn that PQ is the polar of R Hence by the *reciprocal property* of Pole and Polar, PR is the polar of Q, that is to say, the △PQR is self-conjugate with respect to the circle

12 Let P be the point whose polar with respect to a given circle is to be found

Through P draw PAD, PBC cutting the ⊙ at A, D and B, C Let BA, CD intersect at R, and AC, BD at Q. Then, by the last Ex, RQ is the polar of P

If P is an external point, and RQ cuts the circle at T, T′, then clearly PT, PT′ are the required tangents [*Ex* 1, p 332]

13 Let PQR be a triangle self-conjugate with regard to a circle whose centre is O Then since QR is the polar of P, PO is perp to QR [*Def* 2, p 331]

Similarly RO is perp to PQ, and consequently QO is perp to PR [I p 207] That is, O is the orthocentre of the △PQR

14. Let A, P, B, Q be a harmonic range, and O the centre of the given ⊙ Then by the *reciprocal property* of pole and polar, the polars of the points A, P, B, Q are concurrent, since they must all pass through the pole of the line AB And since these polars are respectively perpendicular to OA, OP, OB, OQ, they must form a pencil whose rays contain severally the same angles as the rays of the pencil {O, APBQ} But {O, APBQ} is a harmonic pencil [*Hyp* and *Def* 2, p 327], the pencil formed by the polars is also harmonic

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1 Let TT′ be a common tangent to the two circles, and let their Radical Axis cut TT′ at P Then, by Definition, the tangential distances of the point P to the two ⊙s are equal that is, PT = PT′

2. Let P be any point on the Radical Axis then the four tangents drawn from P to the two circles are equal [*Def*]

Hence a ⊙ described from centre P with any one of these tangents as radius will pass through all four points of contact

And since the radii drawn from P to the points of contact are also tangents to the given circles, the ○ whose centre is P cuts the given ⊙ˢ orthogonally [p 330 *Def*]

3 As in the last example, all tangents drawn from O to the three ○ˢ are equal, a circle from centre O with radius OT will pass through all the points of contact And since the radii of this ○ drawn to the points of contact are also tangents to the given ○ˢ, the ○ whose centre is O cuts the given ○ˢ orthogonally

4. Let the ○ˢ (A), (B), (C) touch one another two and two, and let OT, OT′ be the common tangents of the ○ˢ (A), (B) and (A), (C) at their points of contact

Then since OT and OT′ are tangents to the ○ (A),

$$OT = OT'$$

That is, tangents drawn from O to the ○ˢ (B), (C) are equal

O is a point on the radical axis of the ○ˢ (B), (C)

But the radical axis of two ○ˢ which touch one another is clearly the common tangent at their point of contact

Hence the common tangent to the ○ˢ (B), (C) passes also through O

5 Take the figure of p 208

Since the ∠ˢ BEA, BEC are rt angles, ○ˢ described on AB BC as diamˢ pass through E [*Th* 41],

that is, BE is the common chord of the ○ˢ on AB and BC

Similarly AD and CF are respectively the common chords of the ○ˢ on AB, AC and on BC, CA

Hence O, the point of intersection of the common chords, is the radical centre [p 337].

6 Let C be the centre of the given ○, and A the given fixed point Let DAT be *any* ○ passing through A, and cutting the given ⊙ orthogonally at T Join CA, and produce it, if necessary, to meet the ⊙ DAT at B

Then since the ⊙s cut orthogonally at T, CT is a tangent to the ⊙ DAT [*Th* 46]

$$CB \cdot CA = CT^2 \qquad [Th\ 58]$$

But CA and CT are constant, ∴ CB is constant

∴ B is a fixed point, viz the inverse of A with respect to the given ⊙

7 Since by the last Example all ⊙s which pass through the fixed point A and cut a given ⊙ orthogonally, pass also through a second fixed point B (the inverse of A with regard to the given ⊙), ∴ the locus of their centres is the st line bisecting AB at rt angles

To find this point B, draw *any* radius CT to the given ⊙ describe a ⊙ to pass through A and touch CT at T

Let this ⊙ cut CA at B Then B is the required point, for

$$CA \cdot CB = CT^2 \qquad [Th\ 58]$$

8 Let C be the centre of the given ⊙, and A, D the given points Now by Ex 6 all ⊙s through A cutting the given ⊙ orthogonally must pass through B the inverse point of A with respect to the given ⊙

Determine B as in the last Example Then the ⊙ circumscribed about the △ABD is that required

9 Let P be the centre of any ⊙ which cuts the two given ⊙s orthogonally at T and T′

Then PT = PT′, being radii

Also PT and PT′ are tangents to the given ⊙s, since the ⊙s are cut orthogonally

Hence the locus of P is the radical axis of the two given ⊙s

10 Let C and C′ be the centres of the given ⊙s, and A the given point

Then all ⊙s through A cutting the ⊙ (C) orthogonally pass through B the inverse of A with respect to the ⊙ (C),

and all ⊙s through A cutting the ⊙ (C′) orthogonally pass through B′ the inverse of A with respect to the ⊙ (C′)

Determine the points B and B′ as in the solution to Ex 7

Then the ⊙ about the △ABB′ is that required

Note that by Ex 9 the centre of this ⊙ is on the radical axis of the given ⊙s (C) and (C′)

11. Let A, B be the centres of the two given ⊙s, PQ, PR tangents to them from the given point P Let the Radical Axis cut AB at S

Draw PM, PN perp respectively to AB and the Radical Axis, and bisect AB at O

Then $AP^2 - BP^2 = 2AB \cdot OM$ [*Th* 56 *Ex*]

And $AQ^2 - BR^2 = AS^2 - SB^2$ [*Ex* 1, p 336]

$= 2AB \cdot OS$ [*Th* 56 *Ex*]

∴, by subtraction,

$$AP^2 - AQ^2 - (BP^2 - BR^2) = 2AB(OM - OS),$$

or $PQ^2 - PR^2 = 2AB \cdot SM = 2AB \cdot PN$

12 Let A, B be the centres of two ⊙s of the system, and let their Radical Axis cut AB at S From P, *any* point in the Radical Axis, draw tangents PQ, PR to the two ⊙s, then ∴ PQ = PR [*Hyp*] From centre P, with radius PQ, describe a ⊙ cutting AB at L, L′ Then it will be proved that L, L′ are fixed points for all positions of P

From S draw tangents ST, ST′ to the two ⊙s

Then $SL^2 = PL^2 - PS^2$ [*Th* 29]

$= PQ^2 - PS^2$

$= PA^2 - QA^2 - PS^2$

$= PS^2 + AS^2 - QA^2 - PS^2$

$= AS^2 - AT^2$

$= ST^2$

But ST is independent of the position of P, ∴ L is a fixed point

Similarly $SL' = ST' = ST = SL$

13 Let the radical axis cut the line of centres at S, and let *any* ⊙ of the system cut the same line at X, Y If ST is the tangent from S to this circle, then by definition ST = SL = SL′, where L, L′ are the limiting points

Also $SL^2 = ST^2 = SX \cdot SY$, [*Th* 58]

∴ L, X, L′, Y form a harmonic range [*Ex* 2, p 324], since S bisects LL′

14 With the notation of the last Ex, since L, X, L′, Y form a harmonic range, the polar of L with regard to any circle of the system which cuts the line of centres at X, Y, must cut this line perpendicularly at L′ [*Ex* 4, p 334] But L′ is a *fixed* point [*Ex* 12], the polar of L for all circles of the system is the same

15 Let O, O′ be the centres of two ⊙s which cut one another orthogonally at T Let AB, a diameter of the ⊙ (O), cut the ⊙ (O′) at P, Q

Then $$OP \cdot OQ = OT^2 = OB^2,$$

A, P, B, Q form a harmonic range [*Ex* 2, p 324]

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1 (i) Since $OP \cdot OP' = k^2$, we have $OP' = \frac{k^2}{OP}$, and similarly for OQ′, OR′

Now OP′, OQ′ OR′ are in H P if $\frac{1}{OP'}, \frac{1}{OQ'}, \frac{1}{OR'}$ are in A P, that is, if $\frac{OP}{k^2}, \frac{OQ}{k^2}, \frac{OR}{k^2}$ are in A P, which is true by hypothesis

(ii) Since OP, OQ, OR are in G P, we have $OQ^2 = OP \cdot OR$

$$\frac{k^4}{OQ'^2} = \frac{k^4}{OP' \cdot OR'},$$ whence $OQ'^2 = OP' \cdot OR'$

OQ′ is the Geometric Mean between OP′, OR′

2 The inverse will be a st line par[1] to the base of the △ [*Ex* 2, p 341]

3 Draw any line through O cutting the ⊙ce at P and P′ Let OT be the tangent from O to the ⊙

Invert with respect to the circle whose centre is O and radius OT Then since $OP \cdot OP' = OT^2$ [*Th* 58], the inverse of P will be P′ that is, the inverse of any point on the ⊙ is another point on the same ⊙ Hence the circle inverts into itself

4 The radius of the orthogonal circle is equal to the tangent from its centre to the given circle Hence, by Ex 3, the given circle inverts into itself

5. Draw any other chord through O cutting the ○ at P and P′

Then since OP . OP′ = OA . OB [*Th* 57] = OA^2, the inverse of P will be P′ and the ○ inverts into itself

[Notice that in this case the inverse pt P′ is taken in OP produced *backwards* This kind of inversion is known as *Negative Inversion*]

6 By Ex 1, p 336, the tangents from any pt O on SP to the two circles are equal Taking this as radius of inversion, we see by Ex 3 that each circle inverts into itself

7 By Ex 2, p 337, the tangents from the Radical Centre of three circles to each of the circles are equal

Taking this point as centre of inversion and the tangent to one of the circles as radius of inversion, we see by Ex 3 that each circle inverts into itself

8 Let the diameter of the ○ which is perp to the given line cut the ○ce in O and Q′, and the line in Q.

Let any other line through O cut the line in P and the ○ce in P′ Then since ∠OPQ′ = a rt ∠ [*Th* 41] and ∠PQQ′ = a rt ∠ [*Const*], the points P, P′, Q′, Q are concyclic, and OQ . OQ′ = OP . OP′ Hence if we take O as centre of inversion and k^2 = OQ . OQ′ the given line and the given ○ will each invert into the other [Notice that k = dist from O to the pt of intersection of the line and circle *Th* 66]

9 Let A, B, C denote the three given circles Invert with respect to any point O on the circle S which cuts each of the circles orthogonally [*Ex* 3, p 339]

Then by Ex 3, p 341 the inverse of A is another ○ A′,

and by Ex 2, p 341 the inverse of S is a st line S′

But by Ex 4, p 342, Cor, the angle between the ○ A′ and the line S′ is the same as the angle between A and S, that is, is a rt angle

Hence the line S′ is a diameter of the ○ A′

Similarly S′ is a diameter of B′ and C′

Thus the centres of A′, B′, C′ are collinear

10 Let O be the centre of inversion and C the centre of the given circle

Then by Ex 1, p 340 the inverse of any diameter is a ○ passing through O, and it must also pass through C′, the inverse of C

Thus the diameters invert into a system of ○s passing through O and C′, that is, a co-axal system

Again since the circle and any diameter cut at rt ∠s, their inverses must also cut at rt ∠s [*Ex* 4, p 342 *Cor*] Hence each ○ of the co-axal system cuts the inverse of the given ○ orthogonally

11 Let O be the centre of inversion and P′, Q′, R′ the inverse of P, Q, R Then, by Ex 1, p 340, the pts P′, Q′, R′ lie on a circle through O

By Ex 5, p 342, we have $PQ=\dfrac{k^2 \cdot P'Q'}{OP \cdot OQ'}$, etc

Hence the given relation becomes

$$\frac{k^2 \cdot P'Q'}{OP' \cdot OQ'}+\frac{k^2 \cdot Q'R'}{OQ' \cdot OR'}=\frac{k^2 \cdot P'R'}{OP' \cdot OR'}$$

which simplifies to

$$P'Q' \cdot OR'+Q'R' \cdot OP'=P'R' \cdot OQ',$$

which is the same as Ptolemy's Theorem on four concyclic points [*Th* 78]

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1. (i) Let D, E, F be the mid-pts of BC, CA, AB respectively and let the perp to BC at D meet EF in A′ Similarly for B′, C′

By Ex 3, p 64, EF is par¹ to BC ; ∠DA′E = a rt ∠

Similarly the ∠FC′E is a rt ∠, and the △s DA′E, FC′E are similar

$$\frac{EA'}{EC'}=\frac{ED}{EF}$$

Similarly $\dfrac{FB'}{A'F}=\dfrac{FE}{FD}$, and $\dfrac{DC'}{DB'}=\dfrac{DF}{DE}$

by multiplication, $\dfrac{EA' \cdot FB' \cdot DC'}{AF \cdot B'D \cdot C'E}=1$

∴ by Ceva's Theorem, FC′, EB′, DA′ are concurrent

(ii) Let the bisectors of the $\angle^s$ A, B, C meet the opposite sides in D, E, F

Then, by Theor 61, we have $\frac{BD}{DC}=\frac{BA}{AC}$, $\frac{CE}{EA}=\frac{CB}{BA}$, $\frac{AF}{FB}=\frac{AC}{CB}$

Hence, by multiplication, $\frac{BD\ CE\ AF}{DC\ EA\ FB}=1$

, by Ceva's Theorem, AD, BE, CF are concurrent

(iii) Let D, E, F be the mid-pts of the sides

Then $BD=DC$, $CE=EA$, $AF=FB$

Hence $BD\ CE\ AF=DC\ EA\ FB$

the medians AD, BE, CF are concurrent

2 Applying Menelaus' Theorem to the transversals PEF, RED, QFD in turn, we obtain

$$\frac{BP}{PC}\ \frac{CE}{EA}\ \frac{AF}{FB}=1,$$

$$\frac{AR}{RB}\ \frac{BD}{DC}\ \frac{CE}{EA}=1,$$

$$\frac{CQ}{QA}\ \frac{AF}{FB}\ \frac{BD}{DC}=1$$

Multiplying the three results together, remembering that $AF=AE$, $CE=CD$, $BD=BF$ [*Th* 47 *Cor*], we have

$$\frac{BP}{PC}\ \frac{CQ}{QA}\ \frac{AR}{RB}=1$$

P, Q, R are collinear [*Menelaus' Theorem*]

3 From the $\triangle ABC$ and the transversal PEF,

$$\frac{BP}{PC}\ \frac{CE}{EA}\ \frac{AF}{FB}=1$$

But $AF=EA$ [*Th* 47], $\frac{BP}{PC}=\frac{FB}{CE}$

Hence $BP\ PC=FB\ CE$

$=BD\ DC$ [*Th* 47, *Cor*]

BC is divided harmonically at P and D

4 $\qquad$ BD DC $= \triangle$ BAD $\triangle$ CAD $\qquad$ [*Th* 70]

$\qquad\qquad = BA^2 \quad CA^2$ $\qquad$ [*Th* 72 and 49]

Similarly $\qquad$ CE EA $= CB^2 \quad BA^2$,

and $\qquad$ AF FB $= AC^2 \quad CB^2$

, by multiplication, BD CE AF = DC EA FB

Hence, by the converse of Menelaus' Theorem, the pts D, E, F are collinear

5 Take the figure and the results of p 213

We have, since AF = AE, and BF = BD, and CD = CE,

$$\frac{AF}{FB} \frac{BD}{DC} \frac{CE}{EA} = 1,$$

AD, BE, CF are concurrent [*Ceva's Theorem*]

6 Let the four st lines EAB, EDC, FDA FCB form the complete quad[1] ABCDEF, and let X, Y, Z be the middle points of the diagonals BD, AC, EF

Then X, Y, Z will be proved collinear

Take P, Q, R the middle points of EA, ED, AD

Then from the $\triangle$ AEC, since P and Y are the middle points of AE, AC,

PY is par[1] to EC, and cuts AD at its middle point R

Similarly PZ is par[1] to AF, and cuts ED at its middle point Q, also QX is par[1] to EB, and cuts AD at R

Hence QX, XR, PZ, ZQ RY, YP are respectively halves of EB, BA, AF, FD, DC, CE

But the sides of the $\triangle$ EAD are cut by the transversal BCF,

$$\frac{EB}{BA} \frac{AF}{FD} \frac{DC}{CE} = 1$$

Hence
$$\frac{QX}{XR} \frac{PZ}{ZQ} \frac{RY}{YP} = 1$$

X, Y, Z, points in the sides of the $\triangle$ PQR, are collinear

[See Rouché et de Comberousse, *Traité de Geometrie*, p 205]

7. Let the four lines EAB, EDC, FBC, FAD form the complete quad[l] ABCDEF. Let AC cut BD in P and EF in Q. Let BD, EF cut in R.

From the $\triangle$ CFE and the point A we have by Ceva's Theorem, $FQ \; ED \; CB = QE . DC \; BF$

$$\frac{FQ}{QE}=\frac{FB}{BC}\;\frac{CD}{DE}$$

From the $\triangle$ CFE and the transversal RBD, we have by Menelaus' Theorem

$$FR \; ED \; CB = RE \; DC \; BF$$

Whence $$\frac{FR}{RE}=\frac{FB}{BC}\;\frac{CD}{DE}$$

$$FQ \; QE = FR \; RE$$

Hence EF is divided harmonically at Q and R

Similarly by taking the $\triangle$ AFC with the pt E and the transversal BPD in turn we can shew that AC is divided harmonically at R and P

8 Let the $\triangle$s ABC, A′B′C′ be co polar, that is, let AA′, BB′, CC′ meet at S then shall they be co-axial, that is X, Y, Z the intersections of BC, B′C′, of CA, C′A′ and of AB, A′B′ shall be collinear

From the $\triangle$ SAB and the transversal A′B′Z,

$$\frac{AZ}{ZB}\;\frac{BB'}{B'S}\;\frac{SA'}{A'A}=1$$

From the $\triangle$ SBC and the transversal B′C′X,

$$\frac{B'S}{BB'}\;\frac{C'C}{SC'}\;\frac{XB}{CX}=1$$

From the $\triangle$ SCA and the transversal C′A′Y,

$$\frac{AA'}{A'S}\;\frac{SC'}{C'C}\;\frac{CY}{YA}=1$$

Multiplying these results we have

$$\frac{AZ}{ZB}\;\frac{XB}{CX}\;\frac{CY}{YA}=1$$

X, Y, Z are collinear

Conversely, let X, Y, Z be collinear, then shall AA′, BB′, CC′ be concurrent

Let BB′, CC′ meet at S

Then the $\triangle^s$ BZB′, CYC′ are co-polar, hence by the first proof they are co-axial, that is, A, A′, S are collinear,

or AA′, BB′, CC′ meet at S

9 Let C_1, C_2, C_3 be the centres of the three $\bigcirc^s$, and r_1, r_2, r_3 their radii Let S_1 and S_1' be respectively the external and internal centres of similitude of the $\bigcirc^s$ (C_2), (C_3), and let S_2, S_2', S_3, S_3' have corresponding meanings

To prove S_1', S_2' and S_3 collinear

By definition we have

$$\frac{C_1S_3}{S_3C_2}=\frac{r_1}{r_2}, \quad \frac{C_2S_1'}{S_1'C_3}=\frac{r_2}{r_3}, \quad \frac{C_3S_2'}{S_2'C_1}=\frac{r_3}{r_1}$$

$$\frac{C_1S_3}{S_3C_2}\cdot\frac{C_2S_1'}{S_1'C_3}\cdot\frac{C_3S_2'}{S_2'C_1}=1$$

Hence from the $\triangle C_1C_2C_3$, the points S_1', S_2', S_3 are collinear [*Menelaus' Theorem, Converse*]

In the same way it may be shewn that each of the ranges of points consisting of one external and two internal centres of similitude are collinear, and also that the three external centres are collinear

PART VI

Page 353

1. Parallel lines must be *co-planar*.

3. The intersections (i) of a sphere by a plane, (ii) of the curved surface of a cylinder by the plane ends, (iii) the curved surface of a right cone by the plane base: these are all circles.

4. (i) *One* perpendicular; (ii) an *infinite number* of perpendiculars.

6. At the given point O make a right angle AOB. If OB rotates about OA as axis, then in *all* positions it will be perp. to OA, and in *one* position it will also be perp. to the direction from which it started. Thus we have three directions each of which is perp. to the other two.

   The second statement follows at once from Theor. 81.

7. Let P be any point on the O^{ce}. Join OP, AP. Then in all positions of P the ∠AOP is a rt. angle [p. 318, *Def.* 8],

$$AP = \sqrt{OP^2 + OA^2} \qquad [Th.\ 29.]$$
$$= \text{constant},$$

   for OA is fixed, and OP is of constant length.

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1. One vertical line; an infinite number of horizontal lines.

2. The crease is perpendicular to each of the two short edges, and therefore perpendicular to the plane on which they rest. [*Th.* 81.]

3. Place the spirit-level along any two *intersecting* lines BC, BD (see Fig., p. 354); then, if these lines are found to be horizontal, the vertical line through B is perp. to both, and therefore perp. to the plane XY in which they lie [*Th.* 81]. That is, the plane XY is horizontal.

   Consider the Fig. on p. 370, in which XY represents a horizontal, and ABCD an inclined plane. Then the line of

section CD is horizontal, and all lines drawn *par^l to* CD in the plane ABCD will also be horizontal Thus in an *inclined* plane there can be a series of *parallel* horizontal lines

4 Let Q be *any* point on the $\bigcirc^{ce}$ Then, as in Ex 7, p 353,

$$PQ=\sqrt{OP^2+OQ^2}=\textit{constant} \text{ (for all positions of Q)}$$
$$=\sqrt{(5\ 6)^2+(4\ 2)^2}=\sqrt{49\ 00}=7\ 0 \text{ cm}$$

5 Whatever be the $\angle AOC$, the $\triangle^s$ POA, POB are congruent,

$$PA=PB \qquad [\textit{Th } 4]$$

Similarly $\qquad PC=PD$

$$PA=\sqrt{OP^2+AO^2}=\sqrt{(2\ 4)^2+(1\ 8)^2}=3\ 0''$$

$$PC=\sqrt{OP^2+CO^2}=\sqrt{(2\ 4)^2+(0\ 7)^2}=2\ 5''$$

6 (i) OP, being perp to the plane ABCD, is perp to the diags AOC, BOD at their common mid-pt O, hence the $\triangle^s$ POA, POB, POC, POD are congruent [*Th* 4]

$$PA=PB=PC=PD$$

(ii) Take X the middle point of AB, then $AX=OX=10$ cm

And $\quad OA^2=AX^2+OX^2$ [*Th* 29]

$$=100+100=200$$

Also $\quad PA=\sqrt{PO^2+OA^2}$

$$=\sqrt{1600+200}=\sqrt{1800}=42\ 4 \text{ cm}$$

(iii) Here $\quad OA=\sqrt{AP^2-OP^2}=\sqrt{85^2-75^2}=40$ cm

and $\quad AX=\dfrac{OA}{\sqrt{2}}$ [*Ex* 13, p 124]

$$=\tfrac{1}{2}\times 40\times\sqrt{2}=28\ 3 \text{ cm}$$

$$AB=56\ 6 \text{ cm}$$

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1 Denote the set squares by AOB, A′O′B′, the right angles being at O and O′

Place the set squares so that O and O′ are at the given point, and the edges OB, O′B′ on the given plane (not in one straight line) then bring together the edges OA, O′A′ A straight rod laid along the coincident edges OA, O′A′ will evidently be perp to the given plane

2 Slide one edge of the set square containing the right angle along BC until the perpendicular edge can be made to pass through A this determines the point D (If the perp edge is not long enough to reach to A, it may be prolonged by means of a straight rod)

Use the set square to draw DE in the plane XY perp to BC

Now slide one edge containing the right angle along DE, until the perpendicular edge can be made to pass through A this determines the point P

Hence the position of the perpendicular AP is found

3. The construction of Prob 4, p 74 must be performed in the plane of ABC, and *on the side of* BC *remote from* A, this would be *inside* the given solid.

In BC mark any two points B and C equidistant from A this can be done with compasses

Bisect BC at right angles by DE this construction (*Prob* 2, p 71) is to be done in the plane XY.

Then AD is perp to BC, for the $\triangle^s$ ADB, ADC are congruent [*Th* 7]

By the same method obtain P the foot of the perp from A on DE Then AP is perp to the plane XY

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1 Consult the Fig of p 360

Let A be the given external point, AC *any* one of the equal obliques, and AB the perp from A on the plane XY

Then B is a fixed point, AC, AB are of constant length, and the $\angle$ ABC is a right angle

$$BC=\sqrt{AC^2-AB^2}=constant \qquad [Th\ 29]$$

the locus of C is a circle whose centre is B

2. (i) The *right-angled* $\triangle^s$ PSA, PSB, PSC are congruent [*Th* 18],

$$SA=SB=SC$$

(ii) Here $AB=\sqrt{AC^2+BC^2}=\sqrt{(3\cdot6)^2+(4\cdot8)^2}=6\cdot0''$

And S must be the middle point of AB [*Prob* 25, p 193],

$$PA=\sqrt{PS^2+SA^2}=\sqrt{4\cdot0^2+3\cdot0^2}=5\cdot0''.$$

3 Placing one end of the rod at the given external point P, lay it successively in any three positions oblique to the plane, meeting the latter at the points A, B, and C

Find S, the circum-centre of the △ ABC Then PS is the required perpendicular

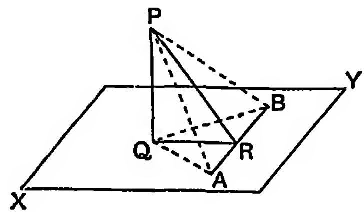
4 Let P be the point at which the three given lines meet From these cut off any three equal lengths, PA, PB, PC Then, as in Ex 3, draw PS perp to the plane ABC

As before, the *right-angled* △s PSA, PSB, PSC are congruent, [*Th* 18]

$$\angle SPA = \angle SPB = \angle SPC$$

5 (i) This is the *Theorem of the Three Perpendiculars* [p 357]

Make RA, RB of any equal lengths



Join QA, QB, also PA, PB

Then from the congruent △s QRA, QRB,

QA = QB [*Th* 4],

and from the congruent △s PQA, PQB,

PA = PB [*Th* 4],

hence from the congruent △s ARP, BRP,

$\angle$ARP = $\angle$BRP [*Th* 7]

PR is perp to AB

(ii) With the same construction as before

The	△s PRA, PRB are congruent [*Th* 4],	PA = PB
Again,	△s PQA, PQB are congruent [*Th* 18],	QA = QB
Lastly,	△s QRA, QRB are congruent [*Th* 7],	$\angle$QRA = $\angle$QRB

QR is perp to AB

6 Here PQ = 1 40 m, and QR = 0 48 m

From the rt angled △ PQR, $PR = \sqrt{PQ^2 + QR^2}$

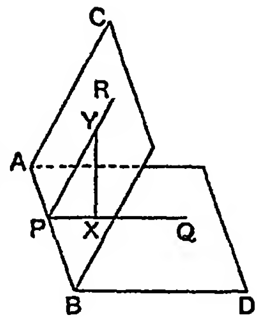
$= 1\ 48$ m

Hence $\cos PRQ = \dfrac{QR}{PR} = \dfrac{48}{148} = 0\ 324$

7. Take X any point in PQ, and let XY, drawn perp to the plane AD, meet the plane BC at Y

To shew that Y *lies in* PR

Join PY Then by the *Theorem of the Three Perpendiculars* [p 357], YP is perp to AB But PR is perp to AB [*Hyp*], and YP is in the plane of AB, PR



Y is a point in PR

XY lies in the plane QPR

8 (i) Let A and B be the two given points, and P *any* point in space equidistant from A and B

Take X, the mid-point of AB, then XP is perp to AB [p 91] Thus every point equidistant from A and B lies in some line perp to AB at its mid-point

But all such lines lie in the plane passing through X perp to AB [p 354, *Cor*]

all points equidistant from A and B lie in this plane

(ii) Let A, B, C be the three points (forming the vertices of a triangle)

Then all points equidistant from A and B lie in the plane perp to AB through its mid-point

And all points equidistant from A and C lie in the plane perp to AC through its mid-point

all points equidistant from A, B, and C must lie in the line of section of the two planes mentioned

By the method of Ex 2, p 361, it may be shewn that the required locus is the perp to the plane of the △ABC through its circum-centre S

(iii) This is worked out in full with a Figure on p 426, the proof there given involving nothing subsequent to Theor 85

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1 Consider the plane of the par[ls] AB, CD

Because AB is equal and par[l] to CD,

AC is equal and par[l] to BD [*Th* 20]

Similarly it may be shewn that CE=DF, and EA=FB

Hence the △s ACE, BDF are congruent [*Th* 7]

2 Let ABCD be a skew quad^l Draw the diag BD

Then the quad^l consists of the $\triangle$'ABD, CBD in two planes of which the line of section is BD

Let X, Y be the mid-points of AB, AD, and let P, Q be the mid-points of CB, CD

Then XY is par^l to BD [*Ex* 2, p 64], and PQ is par^l to BD

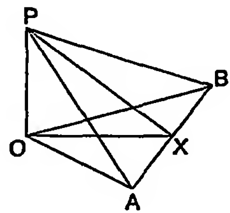
XY is par^l to PQ [*Th* 86]

Similarly XP is par^l to YQ XPQY is a par^m

3 In the $\triangle$ABC, let AX be the perp from A on BC, and let the triangle revolve about BC

Then X is *fixed*, and in all positions AX is at right angles to BC, AX generates a plane [p 354, *Cor*] also AX is of constant length A describes a circle about X as centre

4 The regular hexagon consists of six equilateral triangles consider one of these, namely $\triangle$OAB



Then OX is perp to AB [*Th* 7],

PX is perp to AB [p 357, *Cor*]

(i) $PA=\sqrt{PO^2+OA^2}=\sqrt{92\ 16+16}$

$=\sqrt{108\ 16}$

$=10\ 4$ cm

(ii) $OX=\sqrt{OA^2-AX^2}=\sqrt{16-4}=\sqrt{12}$

$=3\ 5$ cm, to the nearest mm

(iii) $PX=\sqrt{PO^2+OX^2}=\sqrt{92\ 16+12}=\sqrt{104\ 16}$

$=10\ 2$ cm, to the nearest mm

(iv) $\cos OAP=\dfrac{OA}{AP}=\dfrac{4}{10\ 4}=0\ 385$

(v) $\cos OXP=\dfrac{OX}{PX}=\dfrac{3\ 5}{10\ 2}=0\ 339$

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1 AB, CD, being both normal to the given plane, are par^l to one another, and, by hyp, they are equal, AC, BD are equal and par^l [*Th* 20]

Also, since AB is normal to the plane, the ∠ ABD is a right angle. ∴ the fig ABDC is a rectangle

2. Let AB, a normal to the plane, be fixed in position and magnitude, and let C be *any* point whose perp distance from the plane is equal to AB

Then, by Ex 1, AC is perp to AB So that every point on the required locus lies on some line through A perp to AB, and conversely any point on any such line is at the given distance from the plane Now all such lines generate the plane through A perp to AB [p 354, *Cor*], namely the plane through A par[1] to the given plane Hence the required locus is a plane par[1] to the given plane

3. See solution to Ex 8, (1), p 361

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1 If possible, let two planes (P) and (Q) passing through the point A be both par[1] to a third plane (R)

Suppose the planes (P), (Q), and (R) to be cut by a plane (X) which passes through A

Then the lines of section of (X) with (P) and (Q) intersect at A, and, by Theor 89, each of these lines must be par[1] to the section of (X) with (R), which is impossible [p 35]

2 Consider the Fig of p 361 Let the planes XY, PQ be par[1], and let AB be normal to the plane XY Cut the two planes by *any* third plane through AB, then the lines of section are par[1] [*Th* 89] But, by hyp, AB is perp to the section in the plane XY, hence it is also perp to the section in the plane PQ and this section may be *any* line in that plane ∴ AB is normal to the plane PQ.

3 Let the planes (P) and (Q) be each par[1] to the plane (X), and let any line normal to (X) cut that plane at A, and the planes (P) and (Q) at B and C [In the Fig taken, (P) and (Q) are on opposite sides of (X)]

Then because (P) is par[1] to (X), and the line BAC is normal to X, ∴ BAC is also normal to (P) [*Ex* 2]

Similarly BAC is normal to (Q)

∴ the planes (P) and (Q) are parallel [*Th* 88]

4. Let (P) and (Q) be par[l] planes, and let the par[l] lines AB, CD cut (P) at A and C, and (Q) at B and D

Then AC, BD, being the sections made by the plane of AB, CD with (P) and (Q), are par[l] [*Th* 89] the fig ABDC is a par[m] AB=CD

5 *Each* plane of one pair will cut the other pair in two parallel lines Hence there will be *four* parallel sections in all

Page 367

1 Consider the Figure on p 363 Here ABED, CBEF represent the given intersecting planes, which are cut by the two par[l] planes in the lines of section AB, DE, and BC, EF

DE is par[l] to AB, and EF is par[l] to BC [*Th* 89]

the $\angle$DEF=the $\angle$ABC [*Th.* 87]

An exceptional case arises when the par[l] planes are par[l] to BE, the line of section of the intersecting planes In this case the four lines of section are par[l]

2 Let A and B be the two given fixed points Then the locus of points in space equidistant from A and B is the plane perp to AB through its mid point [*Ex* 8, (i), p 361] Hence the required point is that in which the given straight line cuts this plane

The construction is impossible when the given line is parallel to the above mentioned plane

Page 369

1 Let *ab* be the projection of AB on the plane XY [Fig 2, p 368], and let AB be par[l] to the plane XY Then AB, being par[l] to the plane XY, can never meet *ab* which lies in that plane Moreover AB and *ab* are in the same plane AB*ba* [*Th* 92]

AB and *ab* are par[l]

2. Take the Fig of p 369

(i) When AB is par[l] to the plane XY, the fig AB*ba* is a par[m], for A*a* and B*b* are par[l], being both perp to the plane XY, so that *ab*=AB

(ii) When AB is perp to the plane XY, then Bb, Aa become *one line*, namely BA produced, so that a and b coincide, and consequently $ab=0$

(iii) If the angle $\alpha=60°$, the $\triangle$ BAb' becomes half an equilateral triangle, so that Ab' is half of AB

$$ab=\tfrac{1}{2}\ \mathrm{AB}$$

Otherwise the angle $\alpha=60°$,

$$ab=\mathrm{AB}\cos 60°=\mathrm{AB}\times\tfrac{1}{2}$$

3 Consult the Fig of p 360

Here AB is the perp from A on the plane XY, so that BC, BD are the projections on that plane of the obliques AC, AD

Then, if AC = AD, the *right-angled* $\triangle$s ABC, ABD are congruent [*Th* 18], BC = BD

4. Take the Fig of Theor 92 p 368

Suppose another line A′B′ to be drawn par[l] to AB, and let $a'b'$ be the projection of A′B′

Then Bb, B′b' are par[l], both being perp to the plane XY [*Th* 83, *Conv*] Hence the plane of AB, Bb is par[l] to the plane of A′B′, B′b' [*Th* 90], and the projections ab, $a'b'$ are the lines in which these planes are cut by the plane XY

ab, $a'b'$ are par[l] [*Th* 89]

The exceptional case is when AB, A′B′ are perpendicular to the given plane

5 Let BA meet its projection B′A′ produced at P, and let DC meet D′C′ produced at Q.

Then BP, DQ are par[l] [*Hyp*], BB′, DD′ are par[l] [*Th* 83, *Conv*], and PB′, QD′ are par[l] [*Ex* 4],

$$\angle \mathrm{BPB'}=\angle \mathrm{DQD'}=\alpha \text{ (say)} \qquad [Th\ 87]$$

But A′B′ = AB cos α, and C′D′ = CD cos α. [*Th* 92, *Cor* 2]

$$\frac{\mathrm{AB}}{\mathrm{A'B'}}=\frac{1}{\cos\alpha}=\frac{\mathrm{CD}}{\mathrm{C'D'}},$$

or AB : A′B′ = CD : C′D′

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1 (i) is true, (ii) is false

To prove (i), let PQ be par[l] to AB, and let AB be par[l] to the plane XY also let the plane of the par[ls] PQ, AB cut the plane XY in the line CD [Fig p 370]

Then AB is par[l] to CD [*Th* 93, *Conv*],

PQ is par[l] to CD [*Hyp* and *Th* 15],

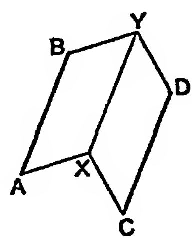
PQ is par[l] to the plane XY, which contains CD [*Th* 93]

2 AB generates a plane par[l] to the given plane XY.

Let *a* be the projection of the fixed point A on the plane XY; and let *ab* be the projection of AB in *any* one of its positions

Then A*ab* is a right angle, and AB *ab* are par[l] [*Th* 93, *Conv*], *a*AB is a right angle and A*a* is a fixed line Hence AB generates a plane [*Th* 82, *Cor*] And this plane is par[l] to the plane XY, for every point in the former lies in some line through A par[l] to the latter

3. Let the two intersecting planes ABYX, CDYX pass respectively through the par[l] lines AB, CD, and let XY be their line of section



Since AB is par[l] to CD it is also par[l] to the plane CDYX which passes through CD [*Th* 93]

Hence the plane ABYX, which passes through AB, cuts the plane CDYX in a line par[l] to AB [*Th* 93, *Conv*]

That is, the line of section XY is par[l] to AB, and consequently also par[l] to CD

4. Let PQ be par[l] to each of the planes ABYX, CDYX, of which the line of section is XY

Through PQ take a plane cutting the plane ABYX in *ab*, and the plane CDYX in *cd*, then *ab*, *cd* are each par[l] to PQ [*Th* 93, *Conv*], and therefore par[l] to one another

Hence, by Ex 3, XY is par[l] to *ab* and *cd*, and therefore to PQ [*Th* 86]

5 Through P draw PX, PY par[l] respectively to AB, CD

Then AB and CD are both par[l] to the plane of XPY, for each is par[l] to a line (PX or PY) in that plane [*Th* 93]

6. Let AB, CD be the two skew lines

In AB take any point P, and in CD take any point Q

Through P draw a line PX par[1] to CD, and through Q draw a line QY par[1] to AB

Then the plane of AB, PX is par[1] to the plane of CD, QY [*Th* 90]

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1 Consider the Fig of Theor 92

Let AB be the given line, and XY the given plane

In AB take any point P, and let Pp be the perp from P on the plane XY.

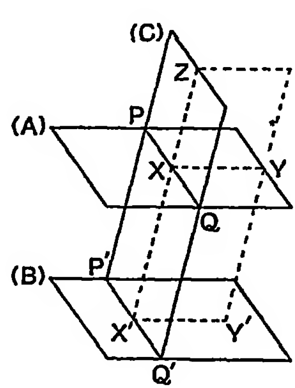
Then the plane through AB, Pp is perp to the plane XY [*Th* 95]

2 Let (X) and (Y) be par[1] planes, and let a line ABC cut (X) at B, and (Y) at C

From a point A in the given line let AP be drawn perp to the plane X, and let AP (or AP produced) cut the plane (Y) at Q Then APQ is normal to the plane (Y) [*Er* 2, p 365], and the angles at which the line ABC cuts the planes (X) and (Y) are measured by the ∠s ABP, ACQ.

Now the plane of ABC, APQ cuts the par[1] planes (X), (Y) in the lines BP, CQ, ∴ BP is par[1] to CQ [*Th* 89], ∴ the corresponding ∠s ABP, ACQ are equal

3 Let the plane (C) cut the par[1] planes (A) and (B) in the lines PQ, P′Q′, then PQ, P′Q′ are par[1] [*Th* 89]

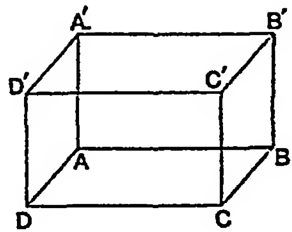


Suppose the three planes (A), (B), (C) to be cut by a plane perp to PQ, and consequently perp to P′Q′ [*Th* 83], and let the sections of this plane with the planes (A), (B), (C) be XY, X′Y′, ZXX′ respectively

Then XZ, XY are perp to PQ [*Def* 8, p 348], and X′Z, X′Y′ are perp to P′Q′, so that the dihedral angles between the planes (A), (C) and (B), (C) are measured by the ∠s ZXY, ZX′Y′. But XY is par[1] to X′Y′ [*Th* 89],

∴ ∠ZXY = ∠ZX′Y′ [*Th* 14]

4 (i) The dihedral angle (α) between the plane ABC′D′ and the floor is measured by the $\angle$DAD′



Now $AD' = \sqrt{(6\,0)^2 + (4\,5)^2} = 7\,5$ m

$$\cos \alpha = \frac{6\,0}{7\,5} = 0\,8000$$

(ii) The dihedral angle (β) between the plane ABCD and the floor is measured by the $\angle$BAB

Now $$AB' = \sqrt{(7\,5)^2 + (4\,5)^2} = 1\,5 \times \sqrt{34}$$

$$\cos \beta = \frac{7\,5}{1\,5 \times \sqrt{34}} = \frac{5\sqrt{34}}{34} = 0\,8574$$

5 Let O be the in-centre of the square, and X the mid-point of AB Then OX is evidently perp^r to AB

Also $OP = 24''$, $OX = 7''$, and $PX = \sqrt{OP^2 + OX^2} = 25''$,

$$\cos OXP = \frac{7}{25} = 0\,2800$$

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1 Call the three planes (A), (B), (C)

Let the section of the planes (A) and (B) be XX′, and the section of the planes (A) and (C) be YY′ Then XX′ and YY′, being both in the plane (A), will in general meet at some point O So that O is a point common to the planes (B) and (C), and therefore lies in the line of section of those planes In other words the three lines of section are concurrent, hence the planes meet at a point

Exceptional cases (i) When the three planes are par^l

(ii) When two planes are par^l, and are cut by the third.

(iii) When the sections of the planes (A) and (B) and of the planes (A) and (C) are *parallel*, in which case all three lines of section are parallel

2 Let ABCD be a skew quadrilateral Join BD, thus dividing the quadrilateral into two triangles *not in the same plane*

Then $\angle A + \angle ABD + \angle ADB = 180°$,

also $\angle C + \angle CBD + \angle CDB = 180°$

Hence $\angle A + \angle C + (\angle ABD + \angle CBD) + (\angle ADB + \angle CDB) = 360°$.

But in the solid angle at B ∠ABD+∠CBD is greater than the ∠ABC, namely the ∠B of the quadrilateral [*Th* 97]
And in the solid angle at D, ∠ADB+∠CDB is greater than the ∠ADC, namely the ∠D of the quadrilateral
Hence ∠A+∠C+∠B+∠D is less than 360°

3. (i) ∠AOX+∠BOX is greater than ∠AOB [*Th* 97]
∠BOX+∠COX is greater than ∠BOC
∠COX+∠AOX is greater than ∠COA
Hence, by addition, twice the sum of the ∠s AOX, BOX, COX is greater than the sum of the ∠s AOB, BOC, COA

(ii) Let OY be the common section of the planes AOB, COX
Then ∠COB+∠BOY is greater than ∠COY [*Th* 97], to each add ∠YOA
Then ∠COB+∠BOA is greater than ∠COY+∠YOA.
But ∠YOA+∠YOX is greater than ∠AOX [*Th.* 97], to each add ∠COX
Then ∠COY+∠YOA is greater than ∠COX+∠AOX
A fortiori ∠COB+∠BOA is greater than ∠COX+∠AOX

(iii) It has been proved that
∠AOX+∠COX is less than ∠AOB+∠BOC,
similarly ∠BOX+∠AOX is less than ∠BOC+∠COA,
and ∠COX+∠BOX is less than ∠COA+∠AOB
Hence, by addition, twice the sum of the ∠s AOX, BOX, COX is less than twice the sum of the ∠s AOB, BOC, COA

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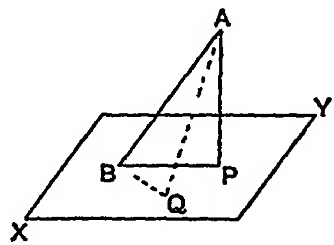
1 Let AB be an oblique, and AP the perp drawn from a point A to a plane XY, so that BP is the projection of AB on that plane



Let BQ be any other line through B in the plane XY
To prove that ∠ABP *is less than* ∠ABQ.
Make BQ equal to BP, and join AQ.
Then the perp AP is less than the oblique AQ [*Th* 85]
And in the △s ABP, ABQ, we have AB, BP equal to AB, BQ respectively, but AP less than AQ,
∠ABP is less than ∠ABQ. [*Th* 19, *Conv*]

2. It will be shewn that the line of greatest slope is perp to the line in which the given plane cuts a horizontal plane

Consider the Fig of p 381

Let C be the given point in the plane BD, which cuts a horizontal plane in the section AB Let CB be perp, and CA oblique to AB, and let CE be perp to the horizontal plane Join EB, EA Then the angles which CB, CA make with the horizontal plane are measured by the $\angle^s$ CBE, CAE

To prove $\angle$CBE *greater than* $\angle$CAE

It may be shewn, as on p 357, *Cor*, that EB is perp to AB

EA is greater than EB [*Th* 12]

From EA cut off EX equal to EB, and join CX

Then the $\triangle^s$ CEB, CEX are congruent [*Th* 4]; so that the $\angle$CBE = the $\angle$CXE

But the ext $\angle$CXE is greater than the int opp $\angle$CAE,

$\angle$CBE is greater than $\angle$CAE

3. Let AB be *any* line through O in the plane XY, let PQ be the perp from P on that plane, and PR the perp on AB Required the locus of R

Join QR, then QR is perp to AB [see solution of Ex 5, (ii), p 361]

Now Q and O being fixed points, and the $\angle$QRO being a right angle, it follows that the locus of R is a circle on QO as diameter

4 (i) Let XYE, XYF be the two planes cutting in the line XY, and let BP, BQ be the lines of section of these two planes with the plane of AP, AQ

Then since AP is perp to the plane XE, the plane of AP, AQ is also perp to the plane XE [*Th* 95]

Similarly the plane of AP, AQ is perp to the plane XF

Hence the plane of AP, AQ, being perp to the planes XE, XF, is perp to XY their line of section [*Th* 96]

(ii) Since XY is perp to the plane of APBQ, BP, BQ are perp to XY, so that the dihedral angle between the given planes is measured by the $\angle$PBQ, which is equal or supplementary to the $\angle$PAQ, since the $\angle^s$ at P and Q are right angles

5 (i) The three points A, B, and D are co-planar Now if AC and BD were co-planar, then C would be in the plane of A B and D, in which case AB and CD would be co-planar, which is contrary to hypothesis.

(ii) As on p 371, *Cor*, it may be shewn that two parallel planes may be passed one through each of two skew lines AB, CD Then the projections of AB, CD on any plane perpendicular to the two parallel planes will be parallel

6. Let AB CD be the two skew lines, and P any point in space

Pass a plane through P and AB, and a second plane through P and CD Since these planes have the point P in common, they cannot be par[l], and hence they intersect in a line through P

This line of section being in the plane of P, AB, and also in the plane of P, CD, will in general intersect both AB and CD

An exceptional case arises when the line of section is par[l] to either AB or CD

7. (i) AO is perp to the plane of OB, OC [*Th* 81], and OX is perp to BC in that plane,

AX is also perp to BC [*Theor of Three Perps*, p 357]

Similarly BY, CZ are perp respectively to CA, AB

XYZ is the pedal triangle of the $\triangle$ ABC

(ii) Produce AP, BP, CP to meet BC, CA, AB respectively at X, Y, Z

Then, as before, AO is perp to the plane of OB, OC,

the plane AOXP is perp to the same plane [*Th* 95]

Again the plane AOXP, which passes through OP is perp to the plane ABC [*Th* 95]

BC, the section of the planes OBC, ABC, is perp to the plane AOXP [*Th* 96]

AX is perp to BC

Similarly BY, CZ are respectively perp to CA, AB,

P is the orthocentre of the $\triangle$ ABC

8. (i) $\sin\theta=\frac{CE}{CA}=\frac{CE}{CB}\cdot\frac{CB}{CA}=\sin\alpha\,.\cos BCA$
$=\sin\alpha\cos\beta$

(ii) $\tan\phi=\frac{EF}{FA}=\frac{CD}{DA}\cdot\frac{DA}{FA}=\tan\beta\sec DAF$
$=\tan\beta\sec\alpha.$

(iii) $\tan\alpha=\frac{CE}{EB}=\frac{CE}{EA}\cdot\frac{EA}{EB}=\tan\theta\sec BEA$
$=\tan\theta\sec\phi.$

(iv) $\sin\beta=\frac{DC}{CA}=\frac{FE}{EA}\cdot\frac{EA}{CA}=\sin\phi\cos\theta$

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2. In the Fig. of p. 382, let Q be the mid-point of OP. Let QN′ be the perp. from Q on the plane of OX, OY, and let OM′, M′N′ be the coords. of N′ in that plane.

Then N′ is the mid-point of ON [*Th.* 92, and *Th.* 22 *Cor.*], and $QN'=\frac{1}{2}PN=5$.

Similarly $OM'=\frac{1}{2}OM=3$, and $N'M'=\frac{1}{2}NM=4$.

∴ required coords. are 3, 4, 5.

3. See p. 385, Art. 8.

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1. The projected area will be the projection of the iron sheet on the horizontal plane.

this area = (area of sheet) × cos 60° = 144 sq. ft. × $\frac{1}{2}$ = 72 sq. ft.

2. (i) $OP=\sqrt{OQ^2+PQ^2}=\sqrt{OA^2+AQ^2+PQ^2}$
$=\sqrt{12^2+9^2+8^2}=\sqrt{289}=17$ cm.

$\cos QOP=\frac{QO}{OP}=\frac{\sqrt{12^2+9^2}}{17}=\frac{15}{17}=0.8824$

Area of fig. OAPR $=OA\times OR=12\times\sqrt{9^2+8^2}=144.49$ sq. cm.

(ii) $\cos\alpha=\frac{OA}{OP}$; $\cos\beta=\frac{OB}{OP}$; $\cos\gamma=\frac{OC}{OP}$

∴ $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\frac{OA^2+OB^2+OC^2}{OP^2}=\frac{OP^2}{OP^2}=1$

$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\left(\frac{12}{17}\right)^2+\left(\frac{9}{17}\right)^2+\left(\frac{8}{17}\right)^2=\frac{289}{289}=1$

(iii) By Theor. 93, the required plane must pass through OP and some line par[l] to BQ. This condition is satisfied by the plane of OP, OA, namely the plane OAPR.

The shortest distance between BQ and OP = the perp. from any point in BQ on the plane OAPR [*Th.* 94], and this is equal to the perp. from B on OR.

Let BX be the perp. from B on OR.

Then $\quad BX \cdot OR = 2\triangle OBR = OB \cdot BR,$

or $\quad BX \cdot \sqrt{b^2+c^2} = bc,$

$$BX = bc/\sqrt{b^2+c^2}$$

3. The cutting plane intersects each pair of opposite faces in par[l] lines [*Th.* 89], hence the lines of section, being coplanar, form a par[m].

4. Consider the Fig. on p. 396, where the prism (ABCDE, A′B′C′D′E′) is cut by par[l] planes in the polygons A$bcde$, A′$b'c'd'e'$.

Now the lateral edges of the prism are all par[l], and, by Theor. 89, the plane of the face DCC′D′ cuts the given par[l] planes in the par[l] lines dc, $d'c'$.

Thus the sides of A$bcde$ are respectively par[l] and equal to the sides of A′$b'c'd'e'$.

Hence the angles of A$bcde$ are respectively equal to the angles of A′$b'c'd'e'$ [*Th.* 87].

∴ the polygons are congruent.

5. Of the six edges of the tetrahedron, two will be equal to a, two to b, two to c, and, on drawing the Fig., it will be seen that each triangular face will have the sides a, b, c; thus all the faces are congruent. Moreover the plane angles at each corner are those opposite respectively to edges a, b and c, that is to say, the angles are equal to those of one triangular face. Hence their sum is 180°.

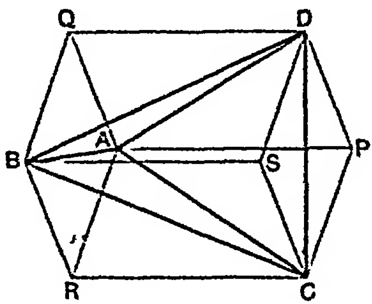
6. The diameter of the circle par[l] to the section of the two given planes will have a projection equal to itself, viz. 10 cm., and this will be the greatest chord of the projected curve.

Projected area = (given area) × cos 45°

$$= \pi \times 5^2 \times \frac{1}{\sqrt{2}} = \pi \times 25 \times \frac{\sqrt{2}}{2}$$

$$= 55.54 \text{ sq. cm.}$$

*** *Examples 7–10 and kindred exercises on the tetrahedron are most simply solved by passing parallel planes through each pair of opposite edges* [Ex. 6, p. 371]. *The six planes so drawn form an enclosing parallelepiped, of which opposite faces have opposite edges of the tetrahedron as diagonals. This is represented in the diagram given below, where* (A BCD) *is the given tetrahedron. (In practice it is easier to draw the enclosing parallelepiped first, and the tetrahedron afterwards.)*



7. In the above diagram let a plane par^l to AC, BD (that is, par^l to the opposite faces ARCP, QBSD of the par^d) cut the edges AB, BC, CD, DA of the tetrahedron in the points E, F, G, H.

Then because AC is par^l to the given cutting plane, and because the plane ACB meets the cutting plane in EF,

EF is par^l to AC. [*Th.* 93, *Cor.*]

Similarly HG is par^l to AC. ∴ EF and HG are themselves par^l [*Th.* [illegible]]

Similarly FG and EH are par^l.

8. In the above diagram, if the tetrahedron is regular, then all the diagonals of the faces of the enclosing parallelepiped are equal. Hence these faces are equal squares, and the parallelepiped is a cube.

Next, the opposite edges AC, BD of the tetrahedron lie in the par^l planes ARCP, QBSD; ∴ the shortest distance between AC, BD is equal to the perpendicular distance between these planes, ∴ = BR.

Thus in the square BRAQ,

$$BR = AB \times \frac{1}{\sqrt{2}} \qquad [Ex\ 13,\ p\ 124]$$

$$= \tfrac{1}{2} \text{ of } AB\ \sqrt{2}$$

$$= \tfrac{1}{2} \text{ (diag of sq on side AB)}$$

9. In the same diagram, if AC, BD are at right angles, then in each of the equal and opp par^{ms} ARCP, QBSD, the two diags are at right angles, hence each of these par^{ms} is equilateral AP = AR

Similarly, if AB, CD are at right angles, each of the par^{ms} AQBR, PDSC is equilateral AQ = AR

AP = AQ = AR

Hence the par^{ms} APDQ, RCSB are equilateral, so that their diags are at right angles

AD and BC are at right angles

10. In the same diagram,

$AC^2 + BD^2$ = the sum of the sqq on the diags of the par^m APCR
= the sum of the sqq on the sides of the par^m APCR [*Ex* 4, p 231]
= $4AP^2$, for the par^m is equilat, [*Ex* 9]
= *constant*, for the par^d has equal edges [*Ex* 9]

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1. (ii) Here $2c(a+b) = 86\ 70$ sq m
and $c = 3\ 40$ m

perimeter $= 2(a+b) = \frac{86\ 70}{3\ 40} = 25\ 50$ m

2 Volume in cubic decimetres = $12\ 5 \times 8\ 0 \times 6\ 5 = 650$

That is, number of litres = 650

Weight in kilograms = $650 \times \frac{4}{5} = 520$ [See p 393]

3 Since 1 hectare = 1000×1000 sq dm, and the rainfall = 6 5 dm, the number of litres required is $1000 \times 1000 \times 6\ 5$, or $6\tfrac{1}{2}$ millions

4 Required weight in kilograms = $12 \times 7\ 5 \times 5 \times 2\ 64$
= 1188

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1 Number of cu yds $= \frac{1325 \times 4}{27} = 196.3$

2 Number of kilograms $= 212.5 \times 15 \times 6.4 = 20400$

3 Volume in cu dm $= 12.8 \times 1.5 \times 0.5 = 9.6$,

required weight $= 9.6 \times 7.8 = 74.88$ kg

4. Volume in cu ft $= 4840 \times 9 \times \frac{7}{2}$

required weight in tons $= \frac{2420 \times 9 \times 7}{2240 \times 16} \times 1000 \times 1.28$

$= 5445$

5 Area of bottom $= (2.50 \times 1.24)$ sq m

Area of two sides $= 2(2.50 \times 1.50)$ sq m

Area of two ends $= 2(1.24 \times 1.50)$ sq m

Total area $= 2(3.74 \times 1.50 + 2.50 \times 1.24)$ sq m

$= 11.22 + 3.1 = 14.32$ sq m

cost $= 14.32 \times 7 = 100.24$ pence $= 8s\ 4d$, to nearest penny

6 Volume in cu dm $= \frac{3}{100} \times 10^2 = 3$

weight in kilograms $= 3 \times 7.14 = 21.42$

7 Here the thickness $= 0.025$ m, and for the inner dimensions *twice* the thickness must be subtracted from the outer. Thus the inner dimensions are

length $= 1.6$ m, width $= 1.2$ m, height $= 0.5$ m

Area of bottom $= (1.6 \times 1.2)$ sq m,

area of two sides $= 2(1.6 \times 0.5)$ sq m,

area of two ends $= 2(1.2 \times 0.5)$ sq m

Total area $= 2(2.8 \times 0.5 + 1.6 \times 1.2)$ sq m.

$= 2.8 + 1.92 = 4.72$ sq m

cost $= 4.72 \times 1.25 = 5.90$ shillings

$= 5s\ 11d$, to the nearest penny

8. (i) If a is the edge of the cube, $6a^2=2.5350$,

$$\text{whence } a=0.65 \text{ m } =65 \text{ cm}$$

(ii) $$\text{Length}=\sqrt[3]{274625}=65 \text{ cm}$$

9. Let $\frac{x}{2}$ = the thickness in centimetres, then the inner dimensions are $12-x$, $10-x$, $8-x$ cm respectively

$$2(12-x)(10-x)+2(12-x)(8-x)+2(10-x)(8-x)=376$$

This equation gives $x=2$

Thus the required thickness is 1 cm

10. Let the dimensions in centimetres be $4a$, $5a$, and $6a$, then $2(20a^2+24a^2+30a^2)=1332$, whence $a=3$

Thus the dimensions are 12, 15, and 18 cm respectively

11. Let the length, breadth and height be a, b and c centimetres respectively,

then $$ab+bc+ca=107,$$
$$ab=42, \quad ac=35$$

These equations give $a=7$, $b=6$, $c=5$

12 Here diagonal $=a\sqrt{3}$ [page 385, Art 8, Cor 2]

$$10=a\sqrt{3}, \text{ whence } a=5.8$$

$$\text{Whole surface}=6a^2=\frac{6\times 100}{3}=200 \text{ sq cm}$$

$$\text{Volume}=a^3=\frac{10\times 10\times 10}{3\sqrt{3}}=\frac{1000\sqrt{3}}{9} \text{ cu cm}$$

$$=\frac{1732.0}{9}=192.4 \text{ cu cm}$$

13. With the notation of p 390, Art 15, we have

$$ab=48, \quad c=3, \quad a^2+b^2+c^2=169,$$
$$a^2+b^2=160, \quad \text{whence} \quad a=12, \; b=4$$

14 Here $$a^2+b^2+c^2=17^2=289,$$
$$2(ab+bc+ca)=552,$$
$$\text{by addition,} \quad (a+b+c)^2=289+552=841$$
$$a+b+c=29$$

15. $$40 \text{ gals per min} = \frac{40}{6\frac{1}{4}} \text{ cu ft per min}$$
$$= \left(\frac{160}{25} \times 60\right) \text{ cu ft per hour}$$
$$\text{the rise per hour} = \frac{160 \times 60}{25 \times 20 \times 16} \text{ feet}$$
$$= 14\ 4 \text{ inches}$$

16. $$\text{Area of base} = (4 \times 15) \text{ sq cm},$$
$$\text{volume} = 4 \times 15 \times 12 = 720 \text{ cu cm}$$
Also $$AB = \sqrt{15^2 + 8^2} = 17 \text{ cm}$$
$$\text{perimeter of base} = (15 + 8 + 17) \text{ cm} = 40 \text{ cm}$$
$$\text{Lateral surface} = (\text{perimeter of base}) \times \text{height}$$
$$= 40 \times 12 = 480 \text{ sq cm}$$

17 The area of the base (found by the method of page 111, or by the formula at the foot of that page) is 36 sq cm

$$\text{volume} = 36 \times 10 = 360 \text{ cu cm}$$
$$\text{Whole surface} = \text{lateral surface} + \text{area of two ends}$$
$$= (36 \times 10 + 2 \times 36) \text{ sq cm} = 432 \text{ sq cm}$$

18 $$\text{Area of base} = \tfrac{1}{2}(13 + 17) \times 8 = 120 \text{ sq cm}, \qquad [Th\ 28]$$
$$\text{volume} = 120 \times 100 = 12000 \text{ cu cm}$$

19. Consider the quantity of sand lying on a strip of ground 4 ft wide and extending for 1 ft of the wall's length This will form a triangular prism whose end is a right-angled triangle The sides containing the right angle are 4 ft and $\frac{4}{\sqrt{3}}$ ft respectively [See *Ex* 14, p 124]

$$\text{Hence volume} = \left(\frac{1}{2} \times 4 \times \frac{4}{\sqrt{3}} \times 1\right) \text{ cu ft}$$
$$= \frac{8\sqrt{3}}{3} = \frac{1\ 732 \times 8}{3} \text{ cu ft} = 4\ 6 \text{ cu ft}$$

20 The capacity of trench $= [\tfrac{1}{2}(15 + 9) \times 8 \times 62\tfrac{1}{2}]$ cu ft
$$= 6000 \text{ cu ft}$$
$$\text{Number of gallons} = 6000 \times 6\tfrac{1}{4} = 37500$$
$$\text{Weight in tons} = \frac{6000 \times 1000}{2240 \times 16} = 167, \text{ nearly}$$

21 The volume of coal is that of a cuboid whose base is $\frac{4840 \times 9}{\cos 23^\circ}$ sq ft, and whose height is 14 ft

Hence weight in tons $= \frac{4840 \times 9 \times 14}{0.9205} \times \frac{1}{28}$

$= \frac{21780}{0.9205} = 23661$, nearly

22. The volume of water flowing through the pipe per minute is that of a cuboid whose dimensions are 400, 0.8, 0.8 decimetres respectively

Hence number of litres per min $= 400 \times \frac{8}{10} \times \frac{8}{10}$

$= 256$

Time required for a million litres $= \frac{1000000}{256}$ minutes

$= 65$ hrs. $6\frac{1}{4}$ min

23. Let the lateral surfaces and the volumes of the two prisms be represented by S_1, V_1, and S_2, V_2 respectively.

Then since *lateral surface = (perimeter of base) × height*,

$$\frac{S_1}{S_2} = \frac{(8 \times 6) \times 6}{(6 \times 8) \times 8} = \frac{3}{4}$$

Again, area of hexagonal base $= (4 \times 4\sqrt{3} \times 6)$ sq cm

$$\therefore V_1 = 16\sqrt{3} \times 6 \times 6 \text{ cu. cm}$$

To find the area of the octagon, produce alternate sides so as to form a circumscribing square Let PQ be a side of this square passing through a side AB of the octagon Then it easily follows that

$$PA = BQ = 3\sqrt{2}, \text{ and } PQ = 6 + 6\sqrt{2}$$

Now area of octagon = sq on PQ − area of 4 corner $\triangle$s

$= PQ^2 - 2PA^2$

$= \{(6 + 6\sqrt{2})^2 - 18 \times 2\}$ sq cm

$= 72(1 + \sqrt{2})$ sq cm

$\therefore V_2 = 72(1 + \sqrt{2}) \times 8$ cu cm

$$\frac{V_1}{V_2} = \frac{16\sqrt{3} \times 6 \times 6}{72(1 + \sqrt{2}) \times 8} = \frac{\sqrt{3}}{1 + \sqrt{2}}$$

$= \sqrt{3}(\sqrt{2} - 1) = \sqrt{6} - \sqrt{3} = .717\ldots$

$V_1 : V_2 = 717 : 1000$

24 The volume removed by excavation is that of a prism whose ends are trapeziums with parallel sides equal to 31 20 and 16 80 metres, and height equal to 4 50 metres.

$$\text{Area of trapezium} = \left(\frac{1}{2}\times 48\times\frac{9}{2}\right)\text{ sq m}, = 108 \text{ sq m.}$$

$$\text{number of tons removed} = 108\times 850\times 2\tfrac{1}{4}$$

$$\text{That is, number of days} = \frac{108\times 850\times 9}{4\times 450} = 459$$

Page 401

3. (i) $V=\frac{1}{3}\times 11\times 7\times 12=308$ cu cm

(ii) By the method of page 111, area of $\Delta=84$ sq cm,

$$V=\tfrac{1}{3}\times 84\times 10 \text{ cu cm}$$
$$=280 \text{ cu cm}$$

4 Using the Figure on p 401, we have

$$SP=\sqrt{9^2+4^2}=\sqrt{97}=9\ 85''$$
$$\text{Again } SA=\sqrt{SP^2+PA^2}=\sqrt{97+16}$$
$$=\sqrt{113}=10\ 63''$$

5 Here
$$S=4\times 5\times\sqrt{10^2+5^2}\text{ sq cm}$$
$$=11\ 18\times 20=223\ 6\text{ sq cm}$$
$$V=\tfrac{1}{3}\times 10\times 10^2=333\ 3\text{ cu cm}$$

6 With the Figure on p 401, we have

$$SP^2=17^2-9^2=208$$
$$SO^2=208-144=64,$$
$$SO=8\text{ cm}$$

Also
$$V=\tfrac{1}{3}\times 8\times 18\times 24=1152\text{ cu cm}$$

7 Here $SP=2\ 5''$, $SO=2\ 4''$, $PO=PA=0\ 7''$,

$$\cos SPO=\frac{PO}{PS}=\frac{7}{25}=0\ 28$$

$$\text{Area of projection of side face}=(0\ 7\times 2\ 5)\times\cos SPO$$
$$=0\ 7\times 2\ 5\times\tfrac{7}{25}\text{ sq in}$$
$$=0\ 49\text{ sq in}$$

8 Replacing the square base of figure on p 401 by an equilateral $\triangle$ABC, we have OP$=5\tan 30^\circ$

$$SP^2=25+25\tan^2 30^\circ=25\sec^2 30^\circ=\frac{25\times 4}{3}$$

$$SP=\frac{10}{\sqrt{3}}=\frac{10\sqrt{3}}{3}=5.8 \text{ cm}$$

Area of side face$=$AP$\cdot$SP$=\frac{10\sqrt{3}}{3}\times 5=28.87$ sq cm.

$$\cos SPO=\frac{OP}{SP}=\frac{5}{\sqrt{3}}\times\frac{\sqrt{3}}{10}=\frac{1}{2}$$

9 Here V$=\frac{1}{3}Ah$, where A is the area of the base and h is the height of the pyramid

Also $\quad A=3\times 3\sqrt{3}\times 6$ sq cm

$=1.732\times 54$ sq cm

$=93.53$ sq cm

$$h=\frac{3V}{A}=\frac{810}{93.53}=8.7 \text{ cm}$$

10 Dividing the wedge as described, we have

volume of pyramid$=\frac{1}{3}(a-c)\times b\times h$;

volume of prism $=\frac{1}{2}bhc$,

hence, by addition,

volume of wedge $=\frac{hb}{6}(2a+c)$

11. Let FG be the perp from F on the base, and FP the perp on BC Then FP meets BC at its middle point P

Slant surface of wedge$=2$(trapezium EB$+$triangle FBC)

Now $FP^2=FG^2+GP^2=h^2+\left(\frac{a-c}{2}\right)^2=\frac{1}{4}\{4h^2+(a-c)^2\}$

area of $\triangle FBC=\frac{1}{2}b\times\frac{1}{2}\sqrt{4h^2+(a-c)^2}$,

area of trapezium EB$=\frac{1}{2}(a+c)\sqrt{h^2+\frac{b^2}{4}}=\frac{1}{4}(a+c)\sqrt{4h^2+b^2}$

Whence, by addition, we obtain the required result

12 Dividing the frustum as described, let the section through A cut CC′ in P, and BB′ in Q. Let Δ be the area of the base.

Then volume of prism $= a\Delta$

The pyramid has for its base the trapezium PQB′C′, and its height $= \dfrac{2\Delta}{PQ}$

$$\text{volume of pyramid} = \tfrac{1}{3} \times \tfrac{1}{2}(C'P + B'Q) \times PQ \times \frac{2\Delta}{PQ}$$
$$= \tfrac{1}{3}\Delta(c - a + b - a)$$
$$\text{whole volume} = \tfrac{1}{3}\Delta(3a + b + c - 2a)$$
$$= \tfrac{1}{3}\Delta(a + b + c)$$

Page 409

1 The volume $= 8.5 \times 6.0 \times 7.4 = 377.4$ cu. cm.

If the dimensions are each 1 mm. in defect,

$$\text{volume} = 8.4 \times 5.9 \times 7.3 = 361.788 \text{ cu. cm.}$$

the greatest possible error in defect $= 15.612$ cu. cm.,

and the error per cent $= \dfrac{15.612}{361.788} \times 100 = 4.31$

If the dimensions are each 1 mm. in excess,

$$\text{volume} = 8.6 \times 6.1 \times 7.5 = 393.450 \text{ cu. cm.}$$

the greatest possible error in excess $= 16.050$ cu. cm.,

and the error per cent $= \dfrac{16.050}{393.450} \times 100 = 4.08$

2 The length of the plank is 10 ft., being the hypotenuse of a right-angled triangle whose sides are 8 ft. and 6 ft.

$$\text{required weight} = \left(10 \times \frac{15}{12} \times \frac{3}{2} \times \frac{1}{12} \times 56\right) \text{ lbs.}$$
$$= 87\tfrac{1}{2} \text{ lbs.}$$

3 The area of the equil. Δ is $5 \times 5\sqrt{3}$ sq. cm.

$$34.64 = 5 \times 5\sqrt{3} \times \cos\theta,$$

that is
$$\cos\theta = \frac{34.64}{5 \times 5 \times 1.732} = 0.8000$$

Whence, by means of Tables, $\theta = 36^\circ 52'$

4. See figure of p 398, Art 21

Here $SO = 8$ cm, $PB = 2$ cm $PO = 2\sqrt{3}$ cm

$$SP = \sqrt{64+12} = \sqrt{76} = 8\ 718 \text{ cm}$$

Slant surf $= \frac{1}{2}$*(perimeter of base)* $\times$ *(slant height)*

$$= \tfrac{1}{2}\ 24 \times 8\ 718 = 104\ 6 \text{ sq cm}$$

Volume $= \frac{1}{3}$*(area of base)* $\times$ *(perp height)*

$$= \tfrac{1}{3}\ 6 \times 2 \times 2\sqrt{3} \times 8 = 64\sqrt{3} \text{ cu cm.}$$

$$= 110\ 8 \text{ cu cm.}$$

5 Use the figure on p 401

$$SP^2 = SA^2 - AP^2 = 6^2 - 3^2 = 27$$

$$OS = \sqrt{SP^2 - OP^2} = \sqrt{27 - 9} = \sqrt{18} \text{ cm}$$

$$= 4\ 24 \text{ cm.}$$

Volume $= \frac{1}{3} \times 36 \times 4\ 24$ cu cm

$$= 50\ 88 \text{ cu cm}$$

6 Let the points in the order given be denoted by A, A′, B, B′, P, and let AB meet the axis of z in Q.

Then $PQ = \sqrt{12^2 + 9^2} = 15$,

and area of $\triangle PAB = 5 \times 15 = 75$ units of area.

Similarly area of $\triangle APA' = 13 \times 9 = 117$ units of area

Thus the whole slant surf $= 2 \times (75 + 117)$ units of area

$= 384$ units of area.

7 See figure of p 398, Art. 21

Here $OB = 5$ cm. $PB = 2\ 5$ cm $\angle SPO = 60°$

Also $OP = OB \sin 60° = \dfrac{5\sqrt{3}}{2}$ cm

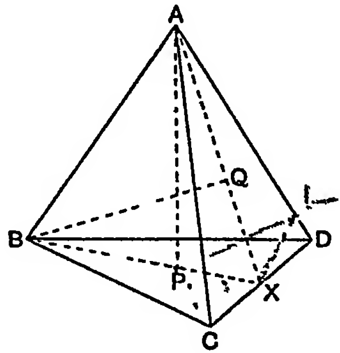
$$SO = OP \tan 60° = \frac{5\sqrt{3}}{2} \times \sqrt{3} = \frac{15}{2} \text{ cm}$$

Volume $= \frac{1}{3}$*(area of base)* $\times$ *height*

$$= \left[\tfrac{1}{3}\left(6 \times 2\ 5 \times \frac{5\sqrt{3}}{2}\right) \times \frac{15}{2}\right] \text{ cu cm}$$

$$= 93\ 75 \times 1\ 732 = 162\ 4 \text{ cu cm}$$

*** *In Examples 8–12 reference will be made to the following diagram*



8\. Let AP be perp to base BCD of a regular tetrahedron (A, BCD). Join BP, CP, DP, and produce them to meet the sides of the base in X, Y, Z.

Then from the rt angled △s APB, APC, APD, we have

$$PB = PC = PD \qquad [Th\ 18.]$$

And from the △s PBC, PBD, the ∠PBC = the ∠PBD [*Th* 7.]

Lastly from the △s XBC, XBD, we have XC = XD [*Th* 4.]

Hence BX is a median of the base: similarly CY, DZ are medians, and P divides each of them in the ratio 2 : 1.
[p 97, III *Cor.*]

9\. Let AP, BQ be the perps from A and B on the opp faces of the tetrahedron: then BP, AQ meet when produced at X the mid-point of CD. [*Ex* 8.]

Hence AP, BQ, AX, BX are coplanar, so that a line PL, drawn par¹ to BQ, must meet AX, and be perp to the face ACD.
[*Th* 83.]

Thus from the similar △s BQX, PLX,

$$BQ : PL = BX : PX = 3 : 1 \qquad [p\ 97,\ Cor.]$$

or,
$$AP : PL = 3 : 1.$$

10\. The faces of a regular tetrahedron are equal *equilateral* △s.

Thus in the above diagram AX = BX.

Now, $BX^2 = BD^2 - DX^2$ [*Th* 29.]

$= 4m^2 - m^2 = 3m^2$

And $\qquad PX=\frac{1}{3}BX, \qquad PX^2=\frac{1}{9}\ BX^2=\frac{m^2}{3}$

Lastly $\qquad AP^2=AX^2-PX^2,$ [Th 29]

or, $\qquad p^2=3m^2-\frac{m^2}{3}$

$\qquad 3p^2=8m^2$

11 (i) With the same Fig as before, we have, as in the last Ex,

$$BX^2=3m^2, \qquad BX=m\sqrt{3}$$

Now *area of face* BCD$=\frac{1}{2}$ CD BX

$$=m^2\sqrt{3}$$

whole surface=sum of four equal faces$=4m^2\sqrt{3}$

(ii) *Volume*$=\frac{1}{3}$ (area of base BCD)$\times$AP

$$=\frac{1}{3}\ m^2\sqrt{3}\times\sqrt{\frac{8m^2}{3}} \qquad [Ex\ 10]$$

$$=\frac{2}{3}\ m^3\sqrt{2}$$

12 Since AX and BX are both perp to CD, the dihedral angle between the faces ACD, BCD is measured by the $\angle$AXB

Now $\qquad \cos AXB=\frac{PX}{AX}=\frac{1}{3}=0\ 3333,$

$\qquad \angle AXB=70°\ 32'$ nearly, from Tables

13 (i) Consult the Fig on p 384

The fig BA′D′C is a par^m [p 384, Art 6],

$\therefore BD'^2+CA'^2=BC^2+CD'^2+D'A'^2+A'B^2$ [Ex 4, p 231]

Similarly $AC'^2+DB'^2=AD^2+DC'^2+C'B'^2+B'A^2$

By addition, sum of sqq on four diags of par^d

$$=AD^2+BC^2+B'C'^2+A'D'^2+(CD'^2+DC'^2)+(A'B^2+B'A^2) \qquad (a)$$

But from the par^m CDD′C′,

$$CD'^2+DC'^2=DC^2+CC'^2+C'D'^2+D'D^2, \qquad (b)$$

and from the par^m ABB′A′,

$$A'B^2+B'A^2=AB^2+BB'^2+B'A'^2+A'A^2 \qquad (c)$$

Substituting in (a) from (b) and (c) we have the required result

(ii) Take the Fig given on p 246 of the *Key*

Let (A, BCD) be a tetrahedron Pass pairs of par[l] planes through the opposite edges, thus enclosing the tetrahedron in the parallelepiped (BRCS, QAPD), of which opp faces have opp edges of the tetrahedron as diagonals

Now the line which joins the mid-points of AC, BD (that is, the common perp to AC, BD) is equal to AQ, a common perp to the opp faces ARCP, QBSD of the par[d]

Similarly the line joining the mid-points of AB, CD, and that joining the mid points of AD, BC are respectively equal to AP and AR

Now $\quad AC^2+BD^2=$ sum of sqq on diags of par[m] ARCP

$$=2[AP^2+AR^2] \qquad [Ex\ 4,\ p\ 231]$$

Similarly $AB^2+CD^2=2[AR^2+AQ^2]$,

and $\quad AD^2+BC^2=2[AP^2+AQ^2]$

By addition, sum of sqq on edges of tetrahedron

$$=4[AP^2+AQ^2+AR^2]$$

14. In the tetrahedron (A, BCD) let the plane ABE, which bisects the dihedral angle between the planes ABC, ABD cut CD at E Through B take a plane perp to AB, and draw CF, EG, DH perp to this plane, so that these lines are co-planar and par[l] to AB and to one another

From C and D draw CK, DL perp to AB,

then $\qquad CK=FB$, and $DL=HB$.

Now $\quad \dfrac{CE}{ED}=\dfrac{FG}{GH}\,[Th\ 22]=\dfrac{FB}{HB}\,[Th\ 61]=\dfrac{CK}{DL}=\dfrac{\triangle CAB}{\triangle DAB}$

15 (i) The pyramid (O, ABC) may be regarded as (C, OAB), that is, as having the vertex C and standing on the base OAB

Hence vol of pyr[d] $=\frac{1}{3}$(area of $\triangle OAB)\times OC$

$$=\frac{1}{3}\ \frac{1}{2}a^2\ a$$

$$=\frac{1}{6}a^3$$

(ii) Draw OP perp to AB, and join CP, then CP is also perp to AB [*Theor of Three Perp'*, p 357]

Then $OP = OA \cos 45° = \frac{a}{\sqrt{2}}$, and $AB = a\sqrt{2}$

$$PC = \sqrt{OP^2 + OC^2} = \sqrt{\frac{a^2}{2} + a^2} = a\sqrt{\frac{3}{2}}$$

Area of $\triangle ABC = \frac{1}{2}\ AB . PC = \frac{1}{2}\ a\sqrt{2}\ a\sqrt{\frac{3}{2}} = \frac{1}{2}a^2\sqrt{3}$

(iii) If p denotes the perp from O on plane ABC,

then vol of pyr^d (O, ABC) $= \frac{1}{3}$(area of $\triangle$ ABC) p,

that is, $$\frac{a^3}{6} = \frac{1}{3}\ \frac{1}{2}a^2\sqrt{3}\ p \qquad [(i) \text{ and } (ii)]$$

$$p = \frac{a}{\sqrt{3}} = \frac{a\sqrt{3}}{3}$$

16 (i) Vol of pyr^d (O, ABC) $= \frac{1}{3}$(area of $\triangle$ OAB) $\times$ OC

$$= \frac{1}{3}\ \frac{1}{2}ab\ c = \frac{1}{6}abc$$

(ii) Draw OP perp to AB, then CP is perp to AB

Now OP AB $= 2\triangle AOB = ab$

$$OP = \frac{ab}{AB}$$

And $$PC = \sqrt{OP^2 + OC^2} = \sqrt{\frac{a^2b^2}{AB^2} + c^2} = \frac{\sqrt{a^2b^2 + c^2\ AB^2}}{AB}$$

$$\triangle ABC = \frac{1}{2}AB\ \ PC = \frac{1}{2}\sqrt{a^2b^2 + c^2AB^2}$$

$$= \frac{1}{2}\sqrt{a^2b^2 + c^2(a^2 + b^2)}$$

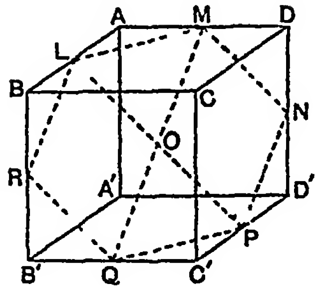
(iii) If p denotes the perp from O on plane ABC,

then vol of pyr^d (O, ABC) $= \frac{1}{3}\ \triangle ABC\ p$

that is, $$\frac{1}{6}abc = \frac{1}{3}\ \frac{1}{2}\sqrt{a^2b^2 + b^2c^2 + c^2a^2} \times p \qquad [(i) \text{ and } (ii)]$$

$$p = abc/\sqrt{a^2b^2 + b^2c^2 + c^2a^2}$$

17. Let (ABCD, A′B′C′D′) be a cube of which O is the centre, and L, M the mid-points of two adjacent edges AB, AD.

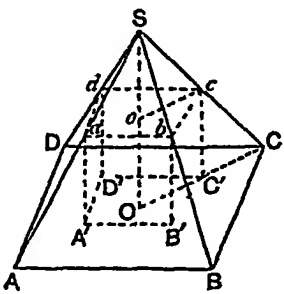


Take a plane through L, M, and O.

This plane will evidently cut the edges C′D′, C′B′ at their mid-points P and Q, and will be seen by the symmetry of the figure to cut the edges BB′, DD′ at their mid points R and N.

Hence the lines of section of the cutting plane with the faces of the cube form the hexagon LMNPQR, which may easily be proved to be regular.

18. Let (S, ABCD) be the square pyramid, and (A′B′C′D′, *abcd*) the inscribed cube, of which each edge $=x$ inches; let the face *abcd* cut the altitude SO at *o*.



Then $Oo=x$, so that $So=h-x$.

Now from similar $\triangle$s,

$$So : SO = oc : OC$$
$$= bc : BC.$$

That is, $h-x : h = x : a$,

or $a(h-x)=hx$,

hence $x=ah/(a+h)$.

Page 413

1. (i) Curved surface $=2\pi rh=(2\pi\times 3\times 8)$ sq. cm

$$=48\pi=48\times\frac{22}{7} \text{ sq. cm}$$
$$=151 \text{ sq. cm}$$

Volume $=\pi r^2h=(\pi\times 9\times 8)$ cu. cm

$$=72\times\frac{22}{7}=226 \text{ cu. cm}$$

(ii) Curved surface $=(2\pi\times 4.5\times 7.2)$ sq. cm

$$=(3.1416\times 9.0\times 7.2) \text{ sq. cm}$$
$$=204 \text{ sq. cm}$$

Volume $=[\pi\times(4.5)^2\times 7.2]$ cu. cm

$$=458 \text{ cu. cm.}$$

2. Total surface $=2\pi r\,(h+r)=(2\pi\times 4{\cdot}2\times 20{\cdot}0)$ sq cm

$=(3{\cdot}1416\times 40\times 4{\cdot}2)$ sq cm

$=528$ sq cm

3. Volume $=\pi r^2 h=3{\cdot}1416\times(1{\cdot}8)^2\times 12=122$ cu cm

4. The locus is evidently the curved surface of a cylinder whose axis is the given straight line, and whose radius is the given distance

Surface $=(2\pi\times 3{\cdot}5\times 5{\cdot}6)$ sq cm.

$=3{\cdot}1416\times 7{\cdot}0\times 5{\cdot}6=123$ sq cm

5. The external and internal radii are 4 cm and 2 cm. respectively.

Area of the two ends $=2\pi(4^2-2^2)=24\pi$ sq cm

The curved external surface $=(2\pi\times 4\times 12)$ sq cm

.. internal .. $=(2\pi\times \underline{2\times 12})$ sq cm

Thus the whole surface $=24\pi\times 7=528$ sq cm

6. Here $h=\dfrac{V}{\pi r^2}=\dfrac{128{\cdot}2}{\pi\times 4}$ m

$=32{\cdot}05\times\dfrac{1}{\pi}=(32{\cdot}05\times 0{\cdot}31831)$ m

$=10{\cdot}20$ m

7. Let d be the diameter in inches

Then $\dfrac{\pi d^2}{4}\times 1000\times 36=$ vol of the wire in cubic inches

$=1$

$d^2=\dfrac{1}{9000}\times\dfrac{1}{\pi}=\dfrac{0{\cdot}31831}{9000}=\dfrac{0{\cdot}00031831}{9}$

$=0{\cdot}000035\ldots;$

$\therefore\ d=0{\cdot}006$ inches, approximately.

8. Here $2\pi rh=1000$, and $2r=20$ $\quad\therefore\ \pi h=50$ cm

Now $V=\pi hr^2=50\times 100=5000$ cu cm

And $h=\dfrac{50}{\pi}=0{\cdot}31831\times 50=15{\cdot}9$ cm

9 The diameter of the base of cylinder is $\sqrt{16^2+12^2}=20$ inches
Thus $r=10$ in

$$\text{Volume of concrete}=8\tfrac{1}{4}\times\left\{\left(\frac{5}{6}\right)^2\pi-\frac{4}{3}\right\}\text{ cu ft}$$
$$=0\ 8484\times 8\tfrac{1}{4}=6\ 99 \quad \text{cu ft}$$

10 The outer and inner radii are 2 7 cm and 2 3 cm respectively

$$\text{volume}=[\pi\{(2\ 7)^2-(2\ 3)^2\}\times 18\times 100]\text{ cu cm}$$
$$=[\pi\times 5\ 0\times 0\ 4\times 18\times 100]\text{ cu cm}$$
$$=\left[3600\pi\times\frac{1}{1000}\right]\text{ cubic decimetres}$$
$$\text{weight}=(3\ 6\pi\times 7\ 79)\text{ kg}$$
$$\doteq 88\ 1\text{ kg}$$

11 Let the length of the wire be x centimetres,

$$\text{then } x\times\frac{2}{10}=\text{surface of cylinder in sq cm}$$
$$=2\pi\times 5\times 12,$$
$$x=600\pi=1884\ 9,$$
$$\text{thus the required length}\doteq 18\ 85\text{ metres}$$
$$\text{Again, volume of wire}=\left[188\ 5\times\pi\times\left(\frac{1}{100}\right)^2\right]\text{ cu dm}$$
$$\text{weight}=(0\ 01885\times 3\ 1416\times 8\ 88)\text{ kg}$$
$$=0\ 52586\text{ kg}$$
$$=525\ 9\text{ grams}$$

Page 417.

1 In the Figure on p 415, the $\angle CAB=\alpha$.

(i) $S=\pi rl$, also $r=h\tan\alpha$, $l=\dfrac{h}{\cos\alpha}$,

$$S=\frac{\pi h^2\tan\alpha}{\cos\alpha}$$
$$V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi h^3\tan^2\alpha.$$

(ii) $S=\pi rl$; and $l=\dfrac{r}{\sin\alpha}$:

$$\therefore\ S=\frac{\pi r^2}{\sin\alpha}$$

$$V=\frac{1}{3}\pi r^2h \text{ and } h=\frac{r}{\tan\alpha};$$

$$\therefore\ V=\frac{1}{3}\cdot\frac{\pi r^3}{\tan\alpha}.$$

Let h_1, h_2 be the heights of two cones with same vertical angle, and let V_1, V_2 be their volumes, then

$$V_1 : V_2=\frac{1}{3}\pi h_1^3\tan^2\alpha : \frac{1}{3}\pi h_2^3\tan^2\alpha$$
$$=h_1^3 : h_2^3.$$

2. (i) $S=\pi rl=\pi\times6\times10=188.4$ sq. cm

$\therefore$ required surface $=188$ sq. cm

$$V=\tfrac{1}{3}\pi r^2h=\tfrac{1}{3}\pi r^2\sqrt{l^2-r^2}$$
$$=\tfrac{1}{3}\pi\times6^2\times8=301.5 \text{ cu. cm.}$$

$\therefore$ required volume $=302$ cu. cm (approx.)

(ii) $S=\pi r\sqrt{r^2+h^2}=\pi\times1.2\times\sqrt{1.44+12.25}$ sq. cm.
$=\pi\times1.2\times\sqrt{13.69}=(\pi\times1.2\times3.7)$ sq. cm.
$=13.9$ sq. cm

$\therefore$ required surface $=14$ sq. cm.

$V=\frac{1}{3}\pi\times(1.2)^2\times3.5=(\pi\times0.48\times3.5)$ cu. cm.
$=5.3$ cu. cm

3. Whole surface $=\pi r(l+r)$

Also $l=\sqrt{r^2+h^2}=\sqrt{1600+81}=41$ cm.

$S=\pi\times9\times50=1413.7$ sq. cm

4. Here $r^2=(l^2-h^2)=(l+h)(l-h)$
$=9.6\times0.6=5.76$

$V=\frac{1}{3}\pi\times5.76\times4.5=(\pi\times5.76\times1.5)$ cu. cm
$=27.14$ cu. cm

Page 421.

1 By formula (i) of Art 34,

$$\text{slant surface} = \tfrac{1}{2}(4\times 20 + 4\times 4)\times l.$$

Also $\underline{l = \sqrt{15^2+8^2} = 17.}$ how ?

$$\text{slant surface} = \tfrac{1}{2}\times 96\times 17 = 816 \text{ sq cm}$$

2 Curved surface $= [\pi(4+3)\times 5]$ sq cm

$$= 35\times \tfrac{22}{7} = 110 \text{ sq cm}$$

3 By formula (ii) of Art 34,

$$\text{volume} = [\tfrac{1}{3}\times 3\times(8^2+8\times 6+6^2)] \text{ cu cm}$$
$$= 148 \text{ cu cm}$$

4 Curved surface $= [\pi\times(4+1)\times 5]$ sq cm

$$= 25\pi = 25\times \tfrac{22}{7} = 79 \text{ sq cm}$$

$$\text{Volume} = \frac{\pi h}{3}(4^2+4\times 1+1^2), \text{ where } \underline{h=4}$$

$$= \frac{\pi}{3}\times 4\times 21 = 4\times 7\times \tfrac{22}{7} = 88 \text{ cu cm}$$

5 Here $l = \sqrt{(5.6)^2+(3.3)^2} = 6.5$

$$\text{slant surface} = [\tfrac{1}{2}(8\times 4+1.4\times 4)\times 6.5] \text{ sq cm}$$
$$= 18.8\times 6.5 = 122.2 \text{ sq cm}$$

6 The triangular ends are right-angled, thus

$$E_1 = \tfrac{1}{2}\cdot 12\cdot 5 = 30 \text{ sq cm, and } E_2 = \tfrac{1}{2}\cdot 6\cdot \tfrac{5}{2} = \tfrac{15}{2} \text{ sq cm}$$

$$\text{volume} = \tfrac{1}{3}\cdot 8[30+\sqrt{30\times \tfrac{15}{2}}+\tfrac{15}{2}] \text{ cu cm}$$
$$= \tfrac{8}{3}\times(\tfrac{75}{2}+15) = 140 \text{ cu cm}$$

7 Volume of cylinder on base of radius $\tfrac{1}{2}(r_1+r_2)$

$$= \pi\frac{(r_1+r_2)^2}{4}\times h$$

Volume of cone on base of radius $\tfrac{1}{2}(r_1-r_2)$

$$= \frac{\pi}{3}\frac{(r_1-r_2)^2}{4}\times h$$

2. Here $$r^2=\frac{1386}{\pi}=441.14,$$
$$r=21 \text{ ft approx.}$$
$$l=\sqrt{30^2+21^2}=36.6 \text{ ft.}$$
$$\text{Curved surface}=\pi r l=(\pi\times 21\times 36.6)\text{ sq. ft.}$$
$$=2414.6 \text{ sq. ft.}$$
required length of canvas 1 yd. wide
$$=(2415\times \tfrac{1}{3})\text{ ft.}$$
$$=805 \text{ ft.}=268 \text{ yds. } 1 \text{ ft.}$$

3. Here $$\tfrac{1}{3}\pi r^2 h=90.478,$$
$$h=\frac{90.478\times 3}{3.1416\times 16}=5.4 \text{ in.}$$

4. The greatest cone will obviously have its vertex in the face opposite to that in which the base lies. And the base of the cone will be the circle inscribed in the square base of the cube. Thus the cone has a radius of 10 cm. and a height of 20 cm.

Hence the whole surface $=\pi\cdot 10^2+\pi\times 10\times\sqrt{10^2+20^2}$
$$=100\pi(1+\sqrt{5})$$
$$=100\pi\times 3.2361$$
$$=1016.6 \text{ sq. cm.}$$

5. In 1 minute a cylindrical column whose base has a radius of 0.25 cm., and whose height is 1000 cm. enters the vessel. Hence if x is the required time in minutes,
$$x\times\pi\times(0.25)^2\times 1000=\tfrac{1}{3}\cdot\pi\times 20^2\times 24,$$
$$x=\frac{400\times 8}{62.5}=\frac{32000}{625}=51.2,$$
thus the required time is 51 min. 12 secs.

6. By Ex. 1 on p. 417, the surface of a cone is given by the formula
$$S=\frac{\pi h^2\tan\alpha}{\cos\alpha}$$
in the present case if h_1 and S_1 in the cone cut off correspond to h and S in the whole cone,
$$S_1 : S=h_1^2 : h^2.$$
But $$S_1=\tfrac{1}{9}S, \qquad h_1=\tfrac{1}{3}h$$
the segments of h made by the cutting plane are in the ratio 1 : 2.

7. The cone has a base of radius 0 7 cm, and its slant height is 2 5 cm

The whole surface required consists of

(i) the circular top, (ii) the curved surface of the cylinder, (iii) the slant surface of the cone

$$\text{whole surface} = \pi(0\,7)^2 + (2\pi \times 0\,7 \times 2\,4) + (\pi \times 0\,7 \times 2\,5)$$

$$= 0\,7\pi(0\,7 + 4\,8 + 2\,5)$$

$$= 5\,6\pi = 17\,59 \text{ sq cm}$$

8 The radius of the top $= 10$ cm, and when the water is drawn off the height of water left is 1 5 cm

if r is the radius of the surface of water,

$$\frac{r}{10} = \frac{1\,5}{7\,5}, \text{ whence } r = 2$$

Now curved surface of frustum of cone $= \pi(2+10)l$

Also l is the hypotenuse of a rt.-angled $\triangle$ whose sides are 6 cm and 8 cm Thus $l = 10$ cm

surface of vessel exposed $= 120\pi = 376\,99$ sq cm

9. Let v and V be the volumes of oil and water respectively Then $v+\text{V}$ is the volume of the whole outer cone Also the outer and inner cones have bases of radii 5 6 cm and 3 5 cm, and they have the same height

$$v \quad v + \text{V} = (3\,5)^2 \quad (5\,6)^2 = 25 \quad 64$$

$$v \quad \text{V} = 25 \quad 39$$

the *weights* are in the ratio $25 \times 0\,92 \quad 39$,

that is, $23 \quad 39$

10 At each end the surface left will be a circular ring whose outer and inner radii are 4 cm and 3 cm respectively

The two cones have base of radius 3 cm and slant height 5 cm

Curved surface of cylinder $= 2\pi \times 4 \times 10$ sq cm

Surface of two ends $= 2 \times \pi(4^2 - 3^2) = 14\pi$ sq cm

Surface of two cones $= 2 \times \pi \times 3 \times 5 = 30\pi$ sq cm

Thus the whole surface $= 124\pi = 389\,56$ sq cm

Page 427

1 In the Fig of p 424, let O be the common centre of two spheres whose radii are ON and OA, and let ON$=r_1$, and OA$=r_2$

Now the tangent plane to the inner sphere at the point N is perp to ON [Art 39, p 424], and cuts the outer sphere in the circular section QPR, of which N is the centre, and NP a radius,

also $\text{NP}=\sqrt{\text{OP}^2-\text{ON}^2}=\sqrt{r_2^2-r_1^2}=a$ *constant*,

for *any* point N on the surface of the inner sphere

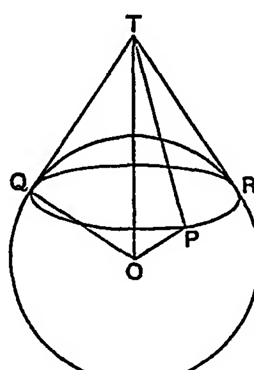
2 Let O be the centre of a sphere of radius r, and X any point on the surface at a given distance a from the fixed point P Then the $\triangle$OXP is of fixed size and shape for all positions of X, and if the $\triangle$OXP revolves about OP as axis, the vertex X passes through all points on the surface at the given distance a from P Now if a triangle revolves about its base, the vertex moves in a plane, and describes a circle [*Ex* 3, p 362]

3 The spheres *touch*, if $r+r'=a$

The spheres *cut*, if $r+r' > a$, and $r \sim r' < a$

If the spheres cut, let O and O′ be their centres, and P a point common to the two surfaces Then, the $\triangle$OO′P being of fixed size and shape, it may be shewn, as in the last Ex, that the locus of P is a circle

4 Let TP be a tangent from the external point T to the sphere whose centre is O Join OT, OP Then the $\angle$OPT is a rt $\angle$ And if the $\triangle$OPT revolves about OT as axis, the point P moves in a circle [*Ex* 3, p 362], and on the surface of the sphere, for OP is its radius



Now in all positions the $\angle$OPT is a rt $\angle$, hence in all positions TP is a tangent to the sphere Hence an *infinite* number of tangent lines can be drawn from T, and the locus of their points of contact is a circle

The tangent TP, as it revolves, generates the curved surface of a cone

5 This Exercise depends on the following extension of Theor 41.

By rotating the Fig of p 164 about the diameter AB it is immediately seen that *a diameter AB of a sphere subtends a right angle at any point on the surface* Hence it follows that *the locus of a point in space at which a fixed line AB subtends a right angle is the surface of a sphere.*

In Ex. 5 let O be the centre of the given sphere, A the given fixed point, and P the centre of the circular section made by any plane through A Then OP is perp to this plane, and the $\angle$APO is a rt angle Hence P moves on the surface of the sphere having the fixed line OA as diameter

If A is *within* or *on* the surface of the given sphere, the required locus is the *whole surface* of the sphere on the diam^r OA, but if A is *outside* the given sphere, the locus of P is that part of the surface of the sphere on OA which falls within the given sphere [Compare Ex 6 p 165]

6 From O draw OA perp to the given plane; and let the circle through Q, P, A cut OA at B

Then OA . OB = OP . OQ [*Th* 58]

= constant, [*Hyp*]

, since OA is fixed, OB is also fixed.

And because A, B, Q, P are concyclic, and the $\angle$ at A is a rt $\angle$.

$\therefore$ the $\angle$Q is a rt $\angle$. [*Th* 41]

Hence the fixed line OB subtends a rt angle at Q, a point in space,

. the locus of Q is the surface of a sphere on OB as diam^r [*Ex* 5]

7. Let (A, BCD) be the given tetrahedron

Consider the planes which bisect the three dihedral angles whose edges are BC, CD, DB These three bisector-planes must meet at some point O [*Ex* 1, p 379], which will be the centre of the inscribed sphere.

It is easily seen that *the locus of points equidistant from two given intersecting planes is the plane which bisects the given dihedral angle*

Hence it follows that the point O is equidistant from the six faces of the tetrahedron, and therefore the centre of the inscribed sphere

The centre of an escribed sphere is found by bisecting *externally* the dihedral angles whose edges are those of one face of the tetrahedron As this process may be performed on each face, there are four possible escribed spheres

8 In the Fig of p 387, suppose (A, BCD) to be a *regular* tetrahedron Then, by symmetry, G is clearly the centre both of the inscribed and circumscribed sphere

and $AG=R$, also $Gq_1=r$

So that $R=3r$ [Art 12, p 387]

Now supposing the $\triangle$ BCD equilateral, each edge being $2a$, we have

$$BX=a\sqrt{3} \text{ [Ex 11, p 124], and } Bq_1=\frac{2}{3}a\sqrt{3} \quad \text{[p 97]}$$

Also $$Aq_1=\sqrt{AB^2-Bq_1^2}=\sqrt{4a^2-\frac{4a^2}{3}}=2a\sqrt{\frac{2}{3}}$$

Now $$AG \text{ (or R)}=\frac{3}{4}Aq_1 \quad \text{[Art 12 p 387]}$$

$$=\frac{3}{4}\,2a\sqrt{\frac{2}{3}}=\frac{a}{2}\sqrt{6}$$

9 It has been shewn [see solution of Ex 5] that the locus of points in space at which a fixed line subtends a right angle is the surface of the sphere having the fixed line as diamr And since in this problem the points are also to be in a given plane, then locus will be the line of section of the plane and the sphere, that is to say, a circle

10 This follows from the fact that *tangent lines drawn to a sphere from an external point are equal*

Let (A, BCD) be a tetrahedron in which a sphere is placed so as to touch the edges AB, AC, AD at the points P, Q, R respectively, and the edges CD, DB, BC at the points X, Y, Z

Then $$AB+CD=(AP+BP)+(CX+DX)$$
$$=AQ+BY+CQ+DY$$
$$=(AQ+CQ)+(BY+DY)$$
$$=AC+BD$$

Page 432

1 (i) $S=4\pi r^2=4\pi\times(2\cdot4)^2=72\cdot4$ sq cm

$$V=\frac{1}{3}\pi r^3=72\cdot4\times\frac{2\cdot4}{3}=72\cdot4\times0\cdot8=57\cdot9 \text{ cu cm}$$

(ii) $S=4\pi r^2=4\pi\times(10\ 5)^2=441\pi=1385\ 44$ sq cm

$V=\frac{4}{3}\pi r^3=1385\ 44\times\frac{10\ 5}{3}=(1385\ 44\times 3\ 5)$ cu cm

$=1849\ 04$ cu cm

2 Cost in shillings $=\frac{1}{2}\times(4\pi\times 6^2)\times\frac{3}{2}=108\pi$

$=339\ 29$

Thus the cost is £16 19*s* 4*d*, to the nearest penny

3 Here $4\pi r^2=\pi\times(1\ 4)^2$, whence $r=0\ 7$ cm

4 Let x be the required number,

then $x\times\frac{4}{3}\pi\ 3^3=\pi\ 2^2\times 45$, whence $x=15$

5. Volume $=\frac{4}{3}\pi(6^3-5^3)=\frac{4}{3}\pi\times 91=381\ 2$ cu cm

6 Whole surface = sum of two hemispherical surfaces + area of circular ring $=2\pi\times 5^2+2\pi\times 4^2+\pi(5^2-4^2)$

$=\pi(50+32+9)=285\ 9$ sq cm

7. Let x cm be the thickness of the tube

Volume of tube $=\pi[5^2-(5-x)^2]\times 4$,

$4\pi[5^2-(5-x)^2]=\frac{4}{3}\pi\ 3^3$,

whence, $(5-x)^2=16$, and $x=1$

8 External curved surface $=\frac{1}{2}\times 4\pi\times 6^2=72\pi$ sq cm

Internal curved surface $=\frac{1}{2}\times 4\pi\times 5^2=50\pi$ sq cm

The flat circular rim $=\pi(6^2-5^2)=11\pi$ sq cm

total surface $=133\pi=417\ 8$ sq cm

Volume of bowl $=\frac{2}{3}\pi(6^3-5^3)$ cu cm

$=\frac{182}{3}\pi=190\ 59$ cu cm

the weight $=(190\ 59\times 8\ 88)$ grams

$=1693\ 95$ grams $=1\ 694$ kg

9. Volume of cylinder $=\pi\times(3\ 5)^2\times(2\pi\times 3\ 5)$ cu cm,

volume of sphere $=\frac{4}{3}\pi(3\ 5)^3$ cu cm

required volume $=(3\ 5)^3\times\pi\times(2\pi-\frac{4}{3})$ cu cm

$=(42\ 87\times 4\ 95\times\pi)$ cu cm

$=212\ 206\pi=666\ 96$ cu cm

10 Here $\quad 4\pi r^2 = \text{curved surface} + \text{surface of two ends}$

$$= 2\pi \times 2 \times 16 + 2 \times \pi \times 2^2$$

$$r^2 = 16 + 2 = 18$$

required radius $= 3\sqrt{2} = 4.24$ cm

11 It is easily seen by similar triangles that at any depth the height of the water will be twice the radius of the surface Let r be this radius in inches when the glass contains 500 drops, then

$$\frac{4}{3}\pi\left(\frac{1}{20}\right)^3 \times 500 = \frac{1}{3}\pi r^2 \times 2r$$

whence $r = \frac{1}{2}$, that is, the height of the water $= 1$ inch

12 Suppose the water rises x centimetres, then the volume of the sphere is equal to that of a cylinder of height x cm and diameter 12 cm

$$\pi \times 6^2 \times x = \tfrac{4}{3}\pi \times 3^3,$$

whence $x = 1$, that is, the water is raised 1 cm

13 The *volumes* of the two spheres are as the cubes of their radii, hence their *weights* will be in the ratio $289 r_1^3 : 64 r_2^3$.

$$\frac{289}{64} \cdot \frac{r_1^3}{r_2^3} = \frac{8}{17};$$

whence $r_1 : r_2 = 8 : 17$

14. Required weight in kilograms

$$= \frac{4}{3}\pi\left(\frac{4}{100}\right)^3 \times 50000 \times 11.35$$

$$= \frac{12.8}{3} \times 11.35 \times \pi = 48.43\pi$$

$$= 152.14$$

15 Required weight in kilograms

$$= \tfrac{4}{3}\pi(6^3 - 4^3) \times 8.88 \times \tfrac{1}{1000}$$

$$= 608\pi \times 2.96 \times \tfrac{1}{1000}$$

$$= 0.608 \times 2.96\pi = 5.6538$$

16 The radius of the wire is $\frac{2}{10^4}$ metres, and the radius of the sphere is $\frac{9}{10^2}$ metres. Let x metres be the length of wire required;

then $$x\times\pi\times\left(\frac{2}{10^4}\right)^2=\frac{4}{3}\pi\times\left(\frac{9}{10^2}\right)^3; \quad \ldots \ldots \quad (1)$$

whence $x=24300$

In the second case, radius of wire $=\frac{95}{100}\times\frac{2}{10^4}$

$=\frac{19}{20}\times\frac{2}{10^4}$

∴ in equation (1) above we have to multiply the coefficient of x by $\left(\frac{19}{20}\right)^2$, and the rest of the equation remains unaltered.

Thus the new length is $\left(\frac{20}{19}\right)^2$ or $\left(1+\frac{39}{361}\right)$ of its former value

Therefore the increase per cent in the length is

$$\frac{39}{361}\times 100, \text{ or } 10.8$$

17. See Figure and notation of p 431, Art 53

Let ANO meet the sphere again in A′ Then we have to find the whole surface and volume of the segment PA′Q

OP $=r=13$ cm NO $=18-r=5$ cm

PN $=r_1=12$ cm AN $=h=8$ cm

Volume of segment PAQ $=\pi h^2\left(r-\frac{h}{3}\right)$

$$=\pi\times 64\times\left(13-\frac{8}{3}\right)=\frac{\pi\times 64\times 31}{3}\text{ cu cm}$$

Volume of sphere $=\frac{4}{3}\pi\times 13^3$ cu cm

∴ vol of segment PA′Q $=\frac{4}{3}\pi(13^3-16\times 31)$ cu cm

$=(4.1888\times 1701)$ cu cm

$=7125.15$ cu cm

Again *whole* surface $=$ area of base $+$ curved surface

$=\pi r_1{}^2+2\pi r h_1$, where $h_1=$ NA′.

$=\pi(144+13\times 18)$ sq cm

$=612\pi=1922.66$ sq cm

18 Here $r=10$ cm, $h=2$ cm

Curved surface $=2\pi\times 10\times 2=40\pi$ sq cm

Area of two ends $=\pi(6^2+8^2)=100\pi$ sq cm,

whole surface $=140\pi=439\ 82$ sq cm

By Art 54,

volume of frustum $=\dfrac{\pi h}{6}(3r_1^2+3r_2^2+h^2)$

$=\dfrac{1}{3}\pi(3\ 8^2+3\ 6^2+2^2)$ cu cm

$=\dfrac{304\pi}{3}=318\ 348$ cu cm

19 Let a perp from the centre O of the sphere meet the plane ends of the zone in P and Q Then PQ=7 cm

If OP$=x$ cm, we have $5^2+(7+x)^2=r^2$

$=12^2+x^2$

From this equation we find $x=5$, and thence $r=13$

surface of zone $=(2\pi\times 13\times 7)$ sq cm

$=182\pi=571\ 77$ sq cm

20 Let O be the centre of the sphere, P the position of the observer's eye Let PR, PR′ be tangents to the sphere Join RR′, meeting OP in Q Then the portion of the sphere visible is the segment whose base is the circle on RR′ as diameter

Then $PR^2=OP^2-OR^2=37^2-12^2$,

whence $PR=35$

Now $OQ\times 37=12^2$, $OQ=\dfrac{144}{37}$

$h=12-\dfrac{144}{37}=\dfrac{12\times 25}{37}$

surface of segment $=\left(2\pi\times\dfrac{12\times 25}{37}\times 12\right)$ sq cm

$=\left(200\pi\times\dfrac{36}{37}\right)$ sq cm

$=611\ 34$ sq cm

21. Using the same Figure and letters as in Ex 20, let OP meet the sphere in T. Then we have to find PT given that the surface of the segment RTR $=\frac{4\pi\times4000^2}{10^6}$ sq mi

$$2\pi\times QT\times4000=\frac{4\pi\times4000^2}{10^6},\quad\text{that is, } QT=\frac{8000}{10^6}=0.008 \text{ mi}$$

Let PT $=x$ miles, and denote QT by h.

Then $$x(x+2r)=PR^2=PQ\cdot PO$$
$$=(x+h)(x+r);$$

whence $$x(r-h)=hr$$

$$x=\frac{hr}{r-h}=\frac{h}{1-\frac{h}{r}}=h\left(1-\frac{h}{r}\right)^{-1}=h, \text{ approx}$$

Thus the required height $=(0.008\times1760\times3)$ ft $=42.24$ ft.

22. Let the perp from B on AO meet AO in P and the circle in B'. Let AO produced meet the circle in A'. Join BA'. Then the required surface is that of a segment of a sphere whose radius is AO, the height of the segment being PA.

$$S=2\pi AP\cdot AO;$$

but from rt-angled $\triangle ABA'$

$$AB^2=AP\cdot AA'=2AP\cdot AO$$
$$S=\pi AB^2$$

23. Denote the volumes of cylinder, hemisphere, and cone by C, H, and C'.

Then obviously, since the heights of the cylinder and cone are each equal to the radius of the base,

$$C=\pi r^2\times r,\quad H=\frac{2}{3}\pi r^3,\quad C'=\frac{1}{3}r\times\pi r^2$$

$$\frac{C}{3}=\frac{H}{2}=\frac{C'}{1}$$

Page 435.

1 From the Fig of p 434, if O' is the centre of the circular section CPB,

we have $$O'P=OP\sin O'OP,$$

or, $$r_1=OP\sin(90^\circ-\theta)=r\cos\theta$$

2 (i) Let l be the length of the equator,

then $l = (2\pi \times 3960)$ mi

$l = 24880$ mi

$\log \pi = 0.4971$
$\log 2 = 0.3010$
$\log 3960 = 3.5977$
$\log l = 4.3958$
antilog $4.3958 = 24880$

(ii) The length of a knot $= \dfrac{24880}{360 \times 60}$ mi

$= 1.152$ mi

(iii) The required length in miles

$= 2\pi r_1$, where $r_1 = 3960 \cos 55°$

$= l \cos 55°$, from (i),

required length $= 14270$ mi

$\log l = 4.3958$
$\log \cos 55° = \bar{1}.7586$
$\log (l \cos 55°) = 4.1544$
antilog $4.1544 = 14270$

(iv) The required distance in miles

$$= \frac{2\pi \cdot 3960 \cos 51° \, 30'}{24}$$

$$= \frac{l \cos 51° \, 30'}{24}$$

required distance $= 645$ mi

$\log l = 4.3958$
$\log \cos 51° \, 30' = \bar{1}.7941$
4.1899
subtract $\log 24 = 1.3802$
2.8097
antilog $2.8097 = 645.2$

3 In the figure of Art 55, let PR be the perp from P to ON Then

(i) Surface of segmental cap

$=$ surface of hemisphere $-$ surface of zone XBCX′

$= 2\pi r^2 - 2\pi r \cdot \mathrm{OR}$

$= 2\pi r^2 - 2\pi r \cdot r \sin \theta$

$= 2\pi r^2 (1 - \sin \theta)$

(ii) By formula (iii) of Art 53,

Volume of segment $= \pi h^2 \left(r - \dfrac{h}{3} \right)$

Also $h = \mathrm{ON} - \mathrm{OR} = r - r \sin \theta$,

$$\begin{aligned} \text{volume} &= \tfrac{1}{3}\pi r^2 (1 - \sin \theta)^2 (3r - h) \\ &= \tfrac{1}{3}\pi r^2 (1 - \sin \theta)^2 (2r + r \sin \theta) \\ &= \tfrac{1}{3}\pi r^3 (1 - \sin \theta)(1 - \sin \theta)(2 + \sin \theta) \\ &= \tfrac{1}{3}\pi r^3 (1 - \sin \theta)(2 - \sin \theta - \sin^2 \theta) \end{aligned}$$

4. The thickness of the zone is evidently $r \sin \theta_1 - r \sin \theta_2$,

surface of zone $= 2\pi r^2 (\sin \theta_1 - \sin \theta_2)$

5. (i) The whole surface of the Earth

$$=4\pi r^2=\pi d^2$$
$$=\pi\times(7922)^2 \text{ sq mi}$$
$\therefore$ surface $=197{,}100{,000}$ sq mi

$\log\pi=0{\cdot}4971$
$2\log d=7{\cdot}7976$
$8{\cdot}2947$
antilog $8{\cdot}2947=197{,}100{,}000$

(ii) Surface of Arctic Cap

$$=2\pi r^2(1-\sin\theta)$$
$\sin\theta=\sin 66\tfrac{1}{2}^\circ=0{\cdot}9171$,
$1-\sin\theta=0{\cdot}0829$.
$\therefore$ surface $=8{,}172{,}000$ sq mi

$\log\pi=0{\cdot}4971$
$\log 2=0{\cdot}3010$
$2\log r=7{\cdot}1956$
$\log 0{\cdot}0829=\bar{2}{\cdot}9186$
$6{\cdot}9123$
antilog $6{\cdot}9123=8{,}172{,}000$

(iii) From the figure in Art. 55,
Surface of Tropical Zone
$=2$(surface of zone XBCX′),
where $\theta=23\tfrac{1}{2}^\circ$;
$$=4\pi r^2\sin 23\tfrac{1}{2}^\circ$$
$$=\pi d^2\sin 23\tfrac{1}{2}^\circ$$
$\therefore$ surface $=78{,}590{,}000$ sq mi

$\log\pi d^2=8{\cdot}2947$, from (i)
$\log\sin 23\tfrac{1}{2}=\bar{1}{\cdot}6007$
$7{\cdot}8954$
antilog $7{\cdot}8954=78{,}590{,}000$

6. In the figure on page 382, join OP and ON.

Then $\angle\text{PON}=\theta$, $\angle\text{MON}=\phi$, $\text{OP}=r$

$x=\text{OM}=\text{ON}\cos\phi$, and $\text{ON}=r\cos\theta$,
$\therefore\ x=r\cos\theta\cos\phi$

$y=\text{NM}=\text{ON}\sin\phi$, and $\text{ON}=r\cos\theta$,
$\therefore\ y=r\cos\theta\sin\phi$

$z=\text{PN}=\text{OP}\sin\theta$
$=r\sin\theta$

Page 440

1. Let r be the radius of the sphere and x a side of the cube, then $\tfrac{4}{3}\pi r^3=x^3$.

$$\therefore\ \frac{r}{x}=\sqrt[3]{\frac{3}{4\pi}}$$
$=0{\cdot}620$, approx.
$r:x=62:100$
$=31:50$

$\log 3=\cdot4771$
$\log 4\pi=1{\cdot}0992$
$3\,|\,\bar{1}{\cdot}3779$
$\bar{1}{\cdot}7926$
antilog $\bar{1}{\cdot}7926=0{\cdot}6203$

$\log\pi=0{\cdot}4971$
$\log 4=0{\cdot}6021$
$1{\cdot}0992$

2 Let x be a side of the cube, d its diagonal, and r the radius of the sphere,

then $4\pi r^2 = 6x^2 = 2d^2$,

$$r = \frac{d}{\sqrt{2\pi}}$$

Hence $r = 23$ cm

$$\begin{array}{r} \log \pi = 0.4971 \\ \log 2 = 0.3010 \\ 2 \,|\, \underline{0.7981} \\ 0.3990 \\ \log 58.4 = 1.7664 \\ \underline{0.3990} \\ 1.3674 \end{array}$$

antilog 1.3674 = 23.20

3 Here $S = \pi r l$, and $l = r\sqrt{\frac{121}{36} + 1} = \frac{r}{6}\sqrt{157}$

$$S = \frac{\pi r^2}{6}\sqrt{157} = \frac{97.6}{6}\sqrt{157},$$

$S = 204$ sq cm

$$\begin{array}{r} \log 97.6 = 1.9894 \\ \log 6 = \underline{0.7782} \\ 1.2112 \\ \tfrac{1}{2}\log 157 = \underline{1.0979} \\ 2.3091 \end{array}$$

antilog 2.3091 = 203.7

Again $\pi r^2 = 97.6$, $r = \frac{\sqrt{97.6}}{\sqrt{\pi}}$ Also $h = \frac{11}{6}r$,

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \cdot \frac{97.6}{\pi} \cdot \frac{11}{6} \cdot \frac{\sqrt{97.6}}{\sqrt{\pi}}$$

$$= \frac{(97.6)^{\frac{3}{2}} \times 11}{18\sqrt{\pi}}$$

$$\begin{array}{r} \log\sqrt{\pi} = .2485 \\ \log 18 = \underline{1.2553} \\ 1.5038 \end{array}$$

$$\begin{array}{r} \log 97.6 = 1.9894 \\ \underline{3} \\ 2 \,|\, \underline{5.9682} \\ 2.9841 \\ \log 11 = \underline{1.0414} \\ 4.0255 \\ \underline{1.5038} \\ 2.5217 \end{array}$$

antilog 2.5217 = 332.4

$V = 332$ cu cm

4 Volume of cylinder $= \pi \times 3^2 \times 6 = 54\pi$ cu ft

Volume of immersed solid $= \left[\left(\frac{2}{3}\pi \times 2^3\right) + \left(\frac{1}{3}\pi \times 2^2 \times 4\right)\right]$ cu ft

$= \frac{\pi}{3}(16 + 16)$ cu ft

$= \frac{32\pi}{3}$ cu ft

vol of water left $= \pi\left(54 - \frac{32}{3}\right) = \frac{130\pi}{3}$ cu ft

$= 136.1$ cu ft

5 Let r be the radius of the base of the cone,

then $$\frac{4}{3}\pi \times (16\ 2)^3 = 5\ 6 \times \frac{1}{3}\pi r^2 \times 70\ 2,$$

$$r^2 = \frac{(16\ 2)^3}{1\ 4 \times 70\ 2}$$

log 1 4 = 0 1461
log 70 2 = 1 8463
1 9924

log 16 2 = 1 2095
3
3 6285
1 9924
2 log r = 1 6361
log r = 0 8180
antilog 0 8180 = 6 577

Thus the radius to the nearest millimetre = 6 6 cm

6. Using the left-hand figure on page 418, we have, with the notation of Art 34,

$h = \mathrm{P}p = 15$ cm, $a_1 = \mathrm{AB} = 40$ cm, $a_2 = ab = 24$ cm,

$$\text{also } l = \sqrt{\frac{225}{4} + 64} = 21\ 9$$

$$\text{Slant surface} = \tfrac{1}{2}\ 4(a_1 + a_2)l$$
$$= (2 \times 64 \times 21\ 9) \text{ sq cm}$$
$$= 2803\ 2 \text{ sq cm}$$
$$\text{Volume} = \tfrac{1}{3}h[40^2 + 40\ 24 + 24^2]$$
$$= 5 \times 3136 = 15680 \text{ cu cm}$$

7. Let h be the height of the cone, and h that of the cylinder, then $h = 217{\cdot}9 - 124\ 3 = 93\ 6''$

Whole volume of tent
$$= \pi r^2 h + \tfrac{1}{3}\pi r^2 h$$
$$= \pi \times 118^2(124\ 3 + 31\ 2)$$
$$= (\pi \times 118^2 \times 155\ 5) \text{ cu inches}$$
$$= \frac{\pi \times 118^2 \times 155\ 5}{1728} \text{ cu ft.}$$

log π = 0 4971
2 log 118 = 4 1438
log 155 5 = 2 1917
6 8326
log 1728 = 3 2375
3 5951
antilog 3 5951 = 3937

volume = 3937 cu ft

8 If r is the radius of the sphere, we have

$$\frac{1}{3}\pi r^2 \times 24\ 6 = \frac{4}{3}\pi \times (7\ 5)^3,$$

$$r^2 = \frac{4 \times (7\ 5)^3}{24\ 6}$$

$$r = 8\ 3 \text{ cm}$$

log 4 = 0 6021
3 log 7 5 = 2 6253
3 2254
log 24 6 = 1 3909
2 | 1·8345 = 2 log r
log r = 0 9172
antilog 0 9172 = 8 264

9 The solid is a double cone

$S = 2\pi rl$, where $r = 3.7 \times \sqrt{3}$, and $l = 7.4$

$= 2\pi \times 3.7 \times \sqrt{3} \times 7.4$

$= 4\pi \times (3.7)^2 \times \sqrt{3}$,

$S = 298$ sq. cm.

$\log \pi = 0.4971$
$\log 4 = 0.6021$
$2 \log 3.7 = 1.1364$
$\frac{1}{2} \log 3 = 0.2385$
2.4741
antilog $2.4741 = 298.0$

$V = \frac{2}{3}\pi r^2 h$, where $h = 3.7$

$= \frac{2}{3}\pi \times (3.7 \times \sqrt{3})^2 \times 3.7$

$= 2\pi \times (3.7)^3$,

$V = 318$ cu. cm.

$\log \pi = 0.4971$
$\log 2 = 0.3010$
$3 \log 3.7 = 1.7046$
2.5027
antilog $2.5027 = 318.2$

10 Let r be the internal radius of the building. Then the volume is made up of a cylinder of height r and radius r, together with a hemisphere of radius r.

$$\pi r^2 \times r + \frac{2}{3}\pi r^3 = 5236,$$

that is $$\frac{5}{3}\pi r^3 = 5236,$$

or $$r^3 = \frac{3 \times 5236}{5 \times 3.1416}$$

$$= \frac{31416}{10 \times 3.1416} = 100,$$

$$r = 10 \text{ ft.}$$

the height $= 2r = 20$ ft.

11 Let R and r be the outer and inner radii of the shell; then

$$V = \frac{4}{3}\pi(R^3 - r^3)$$

$$= \frac{4}{3}\pi(R - r)(R^2 + Rr + r^2)$$

$= \pi \times \frac{4l}{3}(R^2 + Rr + r^2)$, where l is the thickness of the shell,

= vol. of frustum of cone whose height is $4l$, and whose bases are circles of radii R and r.

12 With the Figure on page 415 the inscribed sphere will touch the base at B and have its centre at a point O in AB. Join CO and let $OB = r_1$.

Then since CO bisects the $\angle$ACB.

$$BC : CA = BO : OA;$$

$$\therefore\ r : l = r_1 : h - r_1$$

that is, $$r(h - r_1) = lr_1 \text{ or } rh = r_1(l + r)$$

Now $$\frac{V}{V'} = \frac{\pi r^2 h}{4\pi r_1^3} = \frac{\pi r \times r_1(l + r)}{4\pi r_1^3} = \frac{\pi r(l + r)}{4\pi r_1^2} = \frac{S}{S'}$$

13 Let r be the radius of the base of the cylinder; then the whole solid consists of a sphere of radius r and a cylinder of same radius whose height is $2r$.

$$\frac{4}{3}\pi r^3 + \pi r^2 \times 2r = 464,$$

$$\therefore\ \frac{10}{3}\pi r^3 = 464.$$

$$r^3 = \frac{139 \cdot 2}{\pi}$$

Whole surface = surface of sphere + surface of cylinder,

$$\therefore\ S = 4\pi r^2 + 2\pi r \times 2r = 8\pi r^2$$

$$= 8\pi\left(\frac{139 \cdot 2}{\pi}\right)^{\frac{2}{3}} = 8\pi^{\frac{1}{3}}(139 \cdot 2)^{\frac{2}{3}}.$$

$$\therefore\ \log S = \log 8 + \tfrac{1}{3}\log \pi + \tfrac{2}{3}\log 139 \cdot 2$$

$$= \log 8 + \tfrac{1}{3}(\log \pi + 2\log 139 \cdot 2)$$

$$\therefore\ S = 315 \text{ sq. cm}$$

$\log \pi = 0.4971$
$2\log 139 \cdot 2 = 4.2872$
$3\,)\,4.7843$
1.5948
$\log 8 = 0.9031$
2.4979
antilog $2.4979 = 314.7$

14. In one second the water which flows through the pipe is a cylindrical column whose radius is 15 cm. and length 1·25 m.

Volume of cylinder $= \pi(1.5)^2 \times 12.5$ cubic decimetres

$\therefore$ number of litres per hour $= \pi \times 2.25 \times 12.5 \times 60 \times 60 \times 24$

$= \pi \times 2.25 \times 125 \times 24 \times 360$

$= \pi \times 2.25 \times 1000 \times 1080$

$= 22500 \times 108\pi$

$= 2430000\pi$

$= 7634000$, to the nearest thousand.

15 Volume of cylinder $= (\pi \times 64 \times 4^2)$ cu. cm.

$$\text{weight} = \frac{\pi \times 64 \times 16}{1000} \times 13\cdot6 \text{ kg.}$$
$$= (\pi \times 1\cdot024 \times 13\cdot6) \text{ kg}$$
$$= 43\cdot75 \text{ kg}$$

16 If r is the true value of the radius, the measured value is $r(1 \pm 0\cdot01)$ according as the error of measurement is in excess or defect.

Thus the calculated value of the volume is

$$\tfrac{4}{3}\pi r^3(1\cdot01)^3, \text{ or } \tfrac{4}{3}\pi r^3(0\cdot99)^3;$$

that is $\frac{4}{3}\pi r^3 \times 1\cdot030$ or $\frac{4}{3}\pi r^3 \times 0\cdot970$, approximately, while the true value is $\frac{4}{3}\pi r^3$.

Thus the calculated value may exceed the true value by $\frac{4}{3}\pi r^3 \times \cdot03$, that is by 3 per cent. of itself.

17 Let x = the number of metres, then the volume of the wire is that of a cylinder, $10x$ decimetres in length, on a base of $\frac{0\cdot2}{10}$ or $\frac{2}{10^2}$ decimetres.

weight of wire

$$= \left[10x \times \left(\frac{2}{10^2}\right)^2 \times \pi \times 8\cdot88\right] \text{ kg}$$
$$= \frac{x \times 4\pi \times 8\cdot88}{10^3} \text{ kg}$$
$$\therefore \frac{4\pi \times 8\cdot88}{10^3} x = 593,$$
$$x = \frac{593 \times 10^3}{4\pi \times 8\cdot88}$$

log 4 = 0·6021
log π = 0·4971
log 8·88 = 0·9484
2·0476

log 593 × 10³ = 5·7731
2·0476
3·7255

antilog 3·7255 = 5315·0

Thus the required length = 5315·0 metres.

18 Let n be the required number, then the volume of the shot

$$= \left[n \times \frac{4\pi}{3} \times \left(\frac{1\cdot3}{100}\right)^3\right] \text{ cubic decimetres.}$$
$$\therefore n \times \frac{4\pi}{3}\left(\frac{1\cdot3}{10^2}\right)^3 \times 11\cdot35 = 10\cdot45,$$
$$\therefore n = \frac{10\cdot45 \times 3 \times 10^6}{4\pi \times (1\cdot3)^3 \times 11\cdot35}$$

$$\begin{array}{ll} \log 10.45 = 1.0191 & \log 4 = 0.6021 \\ \log 3 = 0.4771 & \log \pi = 0.4971 \\ \log 10^6 = 6.0 & 3\log 1.3 = 0.3417 \\ \overline{7.4962} & \log 11.35 = 1.0550 \\ 2.4959 & \overline{2.4959} \\ \log n = 5.0003 & \end{array}$$

Thus $n = 100{,}000$, roughly

19. $A = 28.5$ sq in, $B = 78.6$ sq in, $h = 4.5''$

$$\begin{array}{r} 28.5 \\ 47.33 \\ 78.6 \\ \hline 154.43 \\ 1.5 \\ \hline 154.43 \\ 77.22 \\ \hline 231.65 \end{array} \qquad \begin{array}{r} \log A = 1.4548 \\ \log B = 1.8954 \\ \hline 2\,|\,3.3502 \\ \log\sqrt{AB} = 1.6751 \\ \text{antilog } 1.6751 = 47.33 \end{array}$$

Thus the volume $= 232$ cu in

(i) When $A = B$ the solid becomes a cylinder on a base of area A, and with height h, and the formula reduces to hA

(ii) When $B = 0$, the solid becomes a cone on a base of area A, and with height h, and the formula reduces to $\dfrac{hA}{3}$

20. The radius of each tube is $1.25''$ or $\dfrac{1.25}{12}$ ft

$$\text{total heating surface} = \left(2\pi \times \frac{1.25}{12} \times 8 \times 350\right) \text{ sq ft}$$
$$= \left(5 \times 350 \times \frac{\pi}{3}\right) \text{ sq ft}$$
$$= 1832.6 \text{ sq ft}$$

21. Here

$V_1 = \frac{1}{3}\pi \times (8.1)^2 \times 27.5$ cu ft, $V_2 = \frac{1}{3}\pi \times (8.15)^2 \times 27.6$ cu ft

$$\begin{array}{ll} \log \pi = 0.4971 & \log \pi = 0.4971 \\ 2\log 8.1 = 1.8170 & 2\log 8.15 = 1.8224 \\ \log 27.5 = 1.4393 & \log 27.6 = 1.4409 \\ \overline{3.7534} & \overline{3.7604} \\ \log 3 = 0.4771 & \log 3 = 0.4771 \\ \overline{3.2763} & \overline{3.2833} \\ \text{antilog } 3.2763 = 1889 & \text{antilog } 3.2833 = 1920 \end{array}$$

Thus the approximate values of the volume in the two cases are 1889 cu in and 1920 cu in

22 Let d be the diameter then

$$\frac{4}{3}\pi\left(\frac{d}{2}\right)^3=(36.4)^3;$$

$$\therefore\ d^3=\frac{6\times(36.4)^3}{\pi},$$

$$\therefore\ 3\log d=\log 6+3\log 36.4-\log\pi$$

Thus $d=45.16$ cm.

$\log 6=0.7782$
$3\log 36.4=4.6833$
5.4615
$\log\pi=0.4971$
$3\,)\,4.9644=3\log d$
1.6548
antilog $1.6548=45.16$

23 Let d be the diameter of the second sphere; then

$$\frac{\pi\times(5.6)^3}{6}\times1.21 : \frac{\pi d^3}{6}\times0.64=8 : 11;$$

that is, $(5.6)^3\times1.21\times11=d^3\times0.64\times8$

or $(5.6)^3\times(1.1)^3=d^3\times(0.8)^3;$

$$d=\frac{5.6\times1.1}{0.8}=7.7$$

Thus the diameter is 7.7 cm.

24 Let $3h$ be the height of the first cone, and $3r$ the radius of its base; also let V_3 be its volume. The planes of section cut off two cones of heights $2h$ and h respectively on bases of radii $2r$ and r. Let V_2 and V_1 be the volumes of these cones.

Then we have to find the ratios

$$V_1 : V_2-V_1 : V_3-V_2.$$

Now $$\frac{V_1}{r^2h}=\frac{V_2}{(2r)^2\times2h}=\frac{V_3}{(3r)^2\times3h};$$

that is $$\frac{V_1}{1}=\frac{V_2}{8}=\frac{V_3}{27}.$$

$$\therefore\ V_1 : V_2-V_1 : V_3-V_2=1 : 7 : 19.$$

NOTE. The solution may also be very easily obtained by means of formula (1) in Ex. 1 on p. 417.

25 See Ex. 1 on p. 435.

Length of Arctic Circle

$$=2\pi\times3963\times\cos66^\circ30'.$$

Thus the length $=9926$ mi.

$\log 2=0.3010$
$\log\pi=0.4971$
$\log 3963=3.5980$
$\log\cos66^\circ30'=\bar{1}.6007$
3.9968
antilog $3.9968=9926$

Again by Ex. 3 on p. 435, area of required zone

$$=2\pi r^2(\sin\theta_1-\sin\theta_2),$$

where $\theta_1=$ [illegible] and $\theta_2=$ [illegible].

$\sin 65^\circ - \sin 60^\circ =$ [illegible] $-$ [illegible]

$=\cdot 0473$ [illegible]

Thus area $=2\pi r^2\times\cdot 0473$ [illegible]

$=$ [illegible] sq. mi.

[illegible]

26. It is obvious that the diagonals of the cube will be diameters of the sphere. Compare Art. 6, p. 394.

With the notation of Art. 5, Cor. 1, p. 2,

$$d=a\sqrt{3} \text{ and } d=37\cdot 2$$

Required volume

$$=a^3=\left(\frac{37\cdot 2}{\sqrt{3}}\right)^3$$

$$=\frac{(37\cdot 2)^3}{3\sqrt{3}}$$

[illegible]

Thus the volume $=10250$ cu. in.

27. Let V represent the original volume, then at the end of the first day,

volume left $=V(1-\cdot 04)$

$=\cdot 96V=V_1$, say.

After the second day,

volume left $=\cdot 96V_1$

$=(\cdot 96)^2 V$, and so on.

[illegible]

Thus after ten days,

volume left $=(\cdot 96)^{10} V$

$=(\cdot 96)^{10}\times 6^3$

$=143\cdot 7$ cu. ft.;

and 1 cu. ft. weighs 1000 oz.; thus the required weight is 143700 oz.

28 By Ex 11 on p 410, if an edge of the tetrahedron is a centimetres, the volume $=\frac{2}{3}\left(\frac{a}{2}\right)^3\sqrt{2}$ cu cm

$$\therefore \frac{a^3\sqrt{2}}{12}\times 11\cdot 35=10\cdot 70\times 1000,$$

or $$a^3=\frac{12\times 10700}{11\cdot 35\times\sqrt{2}}$$

Thus $a=20\cdot 00$ cm

log 12 = 1·0792
log 10700 = 4·0294
5·1086
1·2055
3) 3·9031 = log a^3
1·3010
antilog 1·3010 = 20·00

log 11·35 = 1·0550
log √2 = 0·1505
1·2055

29. (i) Let ABC be the equilateral Δ revolving about BC Let AD bisect BC at right angles Then the solid generated is a double cone whose height is BD, and whose base has DA for its radius

Now $$BD=\frac{a}{2}, \text{ and } DA=\frac{a\sqrt{3}}{2}$$

$$\text{vol of solid}=2\times\frac{1}{3}\pi\left(\frac{a\sqrt{3}}{2}\right)^2\times\frac{a}{2}$$

$$=\frac{\pi a^3}{4} \qquad (1)$$

Again if G is the centre of the triangle, $DG=\frac{1}{3}DA$. [p 97]

Circumference of circle described by G

$$=2\pi\times\frac{1}{3}\frac{a\sqrt{3}}{2}=\frac{\pi a}{\sqrt{3}}$$

$$=h, \text{ say}$$

Area of triangle $ABC=\frac{a}{2}\times\frac{a\sqrt{3}}{2}$

$$=\Delta, \text{ say}$$

Vol of prism whose base is Δ and height h

$$=\frac{a}{2}\times\frac{a\sqrt{3}}{2}\times\frac{\pi a}{\sqrt{3}}=\frac{\pi a^3}{4},$$

which agrees with formula (1)

(ii) Let the solid be generated by the revolution of the equilateral ΔABC about the line DAE parl to BC Let BD and CE be perps from B and C on this line

Then the required volume is the difference between the cylinder formed by the revolution of DBCE and the two equal cones formed by the $\triangle$s DAE, ADE.

Vol. of cylinder $= \pi \times DE^2 \times BC$.

Vol. of two cones $= 2 \times \frac{1}{3} \times DE^2 \times DA$.

$\therefore$ required volume $= \pi DE^2 \left(BC - \frac{2}{3} DA\right)$

$= \pi$ [illegible]

$= \pi \times$ [illegible]

$=$ [illegible] .... [illegible]

Again if G be the centroid of the triangle AG $= \frac{2}{3} DB =$ [illegible]

Circumference of circle described by G $=$ [illegible]

Area of triangle ABD $=$ [illegible]

Vol. of prism with base [illegible] and height [illegible]

$=$ [illegible]

which agrees with the result.

(3) Let the solid be generated by the revolution of the square ABCD about the line PAQ parallel to the diagonal BD; and let BX, DY be the perpendiculars from B and D on this line.

Then the required volume is twice the difference between the frustum formed by the revolution of the trapezium ACBX and the cone formed by the revolution of the triangle ADX.

Frustum $= \frac{\pi}{3} \cdot AX \cdot (AC^2 + AC \cdot XB + XB^2) =$ [illegible] $\cdot AX^3$,

Cone $= \frac{\pi}{3} \cdot XB^2 \cdot AX$ $=$ [illegible] $\cdot AX^3$;

$\therefore$ required volume $= 2 \times \pi \cdot AX^3$ [illegible] $= 4\pi \cdot AX^3$

$= 4\pi$ [illegible] $=$ [illegible] .... [illegible]

Again if G is the centre of the square,

circumf of circle described by $G = 2\pi \cdot AG$

$$= 2\pi \cdot \frac{a}{\sqrt{2}} = \pi a\sqrt{2}$$

Area of square $= a^2$

volume of solid of revolution $= \pi a\sqrt{2} \times a^2 = \pi a^3\sqrt{2}$,

which agrees with formula (iii)

(iv) Let the solid be generated by the revolution of the hexagon ABCDEF about AB, and let CX, FY be the perps from C and F on AB produced.

Then the required volume

$=$ cylinder formed by revolution of ABDE

$+$ two equal frusta by revolution of BDCX, AEFY

$-$ two equal cones by revolution of Δ^s BCX, AFY

Vol of cylinder $= \pi \cdot AB \cdot BD^2 = \pi a(a\tan 60^\circ)^2 \quad = 3\pi a^3$

Vol of two frusta $= 2 \times \frac{\pi}{3} \cdot BX\{BD^2 + BD \cdot XC + XC^2\} = \frac{7}{4}\pi a^3$

Vol of two cones $= 2 \times \frac{\pi}{3} \cdot BX \cdot CX^2 \quad = \frac{1}{4}\pi a^3$,

required volume $= \pi a^3\left[3 + \frac{7}{4} - \frac{1}{4}\right] = \frac{9}{2}\pi a^3$

This result may be verified as in the former examples

30 Area of circle $= \pi \times \left(\frac{7}{4}\right)^2$ sq in

Circumference of ○ described by centre

$= 2\pi \times 7$ in

required volume

$$= \frac{\pi^2 \times 7^3}{8}$$

$= 423\cdot1$ cu in

$2\log\pi = 0\cdot9942$
$3\log 7 = 2\cdot5353$
$3\cdot5295$
$\log 8 = 0\cdot9031$
$2\cdot6264$
antilog $2\cdot6264 = 423\cdot1$

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